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GLOBAL EQUILIBRIUM STABILITY AND STABLE PERIOD-TWO OSCILLATIONS IN A CAPACITY-BOUNDED INVENTORY ADJUSTMENT MODEL

Robertus Heri Soelistyo Utomo and Hafidh Khoerul Fata *

Department of Mathematics, Faculty of Science and Mathematics, Universitas Diponegoro, Semarang, Indonesia.
* Corresponding Author; Email: hafidz_fata@live.com

ORCID Details
Robertus Heri Soelistyo Utomo: https://orcid.org/0000-0002-1936-5793
Hafidh Khoerul Fata: https://orcid.org/0000-0001-6468-6118

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ABSTRACT

Repeated inventory replenishment can generate alternating stock levels within a fixed storage capacity. We analyze a discrete-time inventory balance with proportional depletion and a bounded sigmoid replenishment rule. The map preserves the normalized capacity interval and has a unique equilibrium. For the centered rule, an explicit sensitivity threshold separates global exponential convergence from equilibrium instability. At the threshold, trajectories converge algebraically at rate $n^{-1/2}$. Above it, the map has a unique orbit of prime period two whose multiplier lies inside the unit circle along the entire branch. The oscillation amplitude grows monotonically, follows a square-root onset law and approaches a finite limit. With full adjustment, the cycle attracts every non-equilibrium trajectory. Numerical calculations illustrate the stability threshold, the critical decay and the period-two branch. The model isolates deterministic feedback under immediate replenishment.

Keywords: Inventory Adjustment; Invariant Interval; Global Stability; Flip Bifurcation; Period-Two Orbit; Critical Convergence.

1 INTRODUCTION

Inventory replenishment links successive decisions: the stock remaining after one period determines the target for the next. This feedback can produce oscillations even under a constant demand assumption. Forrester [1] established stock adjustment as a feedback problem, while Sterman's experiments [2] showed how decision-makers can misread its dynamics. Dejonckheere et al. [3] quantified order variability under alternative replenishment rules, and Hoberg et al. [4] examined the roles of inventory information and control policy. Both lines of work point to the adjustment rule itself as a possible source of persistent variation.

Operational constraints can alter the feedback mechanism. Nonnegative-order policies admit oscillations [5]; finite production capacity [6] and lost-sales rules [7] introduce other nonlinear effects. Wang and Disney [8] summarize how

forecasting and lead-time assumptions affect order amplification. A scalar, capacity-preserving adjustment rule raises a more focused question: whether bounded stock can coexist with an unstable equilibrium and sustained periodic behavior.

Inventory-control studies also differ in what they seek to optimize or measure. Xu et al. [9] designed an optimal-control intervention against order amplification, a quantity defined through orders rather than stock alone. Cui et al. [10] examined proportional order-up-to replenishment with explicit lead times, so the stock response depends on outstanding orders. Standard treatments additionally include costs, service constraints and ordering decisions [11]. These mechanisms are indispensable for operational evaluation, but they complicate the separation of effects attributable to the shape of the replenishment response. We therefore consider immediate replenishment with a bounded nonlinear target and ask how sensitivity controls the transition from a steady stock level to alternating states. The reduced model permits an explicit analytical answer without estimating operational performance.

We combine proportional depletion with a decreasing sigmoid replenishment target. The resulting map preserves the normalized stock interval. For a centered target, shifting coordinates yields an odd map and a scalar equation for the period-two orbit. Difference-equation theory [12] and discrete-time bifurcation analysis [13] motivate the approach; the explicit stability threshold, critical coefficient and orbit multiplier are derived for the chosen feedback rule. Local and global conclusions are kept separate [14].

Our results first establish equilibrium feasibility and uniqueness across target positions. We then analyze the centered policy at and beyond its stability boundary, including critical decay and the stable period-two branch. A full basin result is obtained for complete adjustment, and computations check the formulas.

2 MODEL FORMULATION AND METHODOLOGY

The stock variable is normalized by storage capacity. We specify depletion and replenishment in a one-period balance, then shift the resulting map about the centered target to expose its symmetry. Let x_n denote inventory immediately before the replenishment decision in period n, expressed as a fraction of storage capacity. A one-step accounting relation is

$$x_{n+1} = x_n - d_n + u_n, \qquad n = 0,1,2,\dots, \tag{1}$$

where d_n is stock depletion during the period and u_n is the effective addition to inventory. The following idealizations are adopted:

$$d_n = \delta x_n, \qquad u_n = \frac{\delta}{1+\exp\{\kappa(x_n-\theta)\}}, \tag{2}$$

where $0 < \delta \le 1$, $\kappa > 0$, and $0 < \theta < 1$. The parameter δ is the adjustment or depletion fraction, κ controls replenishment sensitivity, and θ sets the center of the replenishment response. The factor δ bounds each effective addition by $0 < u_n < \delta$. Using the same adjustment factor for depletion and replenishment specifies the simplified stock balance analyzed below.

Substitution gives the discrete-time map

$$x_{n+1} = F_{\delta,\kappa,\theta}(x_n), \qquad F_{\delta,\kappa,\theta}(x) = (1-\delta)x + \frac{\delta}{1+\exp\{\kappa(x-\theta)\}}. \tag{3}$$

The sigmoid target lies strictly between zero and one for every feasible stock. Consequently, (3) preserves the capacity interval without a clipping rule. The same factor scales depletion and replenishment, ensuring this convex-combination structure. Replenishment enters the next stock observation immediately. Demand uncertainty, pipeline inventory, backorders and delivery delays are outside the model. We vary the slope of the sigmoid to determine when increasingly sensitive adjustment produces alternating inventory levels.

The principal analysis takes $\theta = \frac{1}{2}$. Let

$$y_n = x_n - \frac{1}{2}. \tag{4}$$

Using $(1+e^z)^{-1} = \frac{1-\tanh\left(\frac{z}{2}\right)}{2}$, equation (3) becomes

$$y_{n+1} = G(y_n), \qquad G(y) = (1-\delta)y - \frac{\delta}{2}\tanh\left(\frac{\kappa y}{2}\right), \qquad -\frac{1}{2} \le y \le \frac{1}{2}. \tag{5}$$

The function G is smooth and odd. Hereafter a fixed point is globally attractive on the admissible interval when every initial condition in that interval converges to it. Linear asymptotic stability of a periodic orbit refers to the absolute value of its return map derivative being less than one.

Two arguments drive the analysis: monotonicity gives the unique equilibrium, and a uniform derivative bound yields the global contraction regime. The centered map is odd, which reduces period-two existence to a positive scalar root and, together with the monotonicity of the sigmoid, rules out nonsymmetric period-two orbits. At criticality, a cubic expansion determines the decay rate. We then check the analytical results numerically by iterating the map, bracketing the root and evaluating the orbit multiplier. The computations are illustrative; the stability and existence claims follow from the proofs.

3 INVARIANCE AND EQUILIBRIUM STABILITY

Feasibility follows from the bounded response, but the long-run dynamics depend on feedback sensitivity. We begin with a general equilibrium result and then analyze the centered policy.

Proposition 1 (capacity preservation and equilibrium uniqueness). For every $0 < \delta \le 1$, $\kappa > 0$, and $0 < \theta < 1$, the interval $[0,1]$ is positively invariant under (3). The map has exactly one equilibrium in $(0,1)$. For the centered policy $\theta = \frac{1}{2}$, this equilibrium is $x^* = \frac{1}{2}$. The invariance and uniqueness conclusions also hold when the sigmoid target is replaced by any continuous, nonincreasing response $R\colon [0,1] \to (0,1)$.

Proof. If $x \in [0,1]$, then $0 < (1 + \exp\{\kappa(x-\theta)\})^{-1} < 1$. Equation (3) is a convex combination of x and this bounded target, so $F_{\delta,\kappa,\theta}(x) \in [0,1]$. At equilibrium, cancellation of the common factor $\delta > 0$ yields

$$x = \frac{1}{1 + \exp\{\kappa(x-\theta)\}}. \tag{6}$$

The left side is strictly increasing and the right side strictly decreasing on $[0,1]$; their difference is negative at $x = 0$ and positive at $x = 1$. Hence there is exactly one solution in $(0,1)$. For $\theta = \frac{1}{2}$, direct substitution gives $x^* = \frac{1}{2}$. More generally, the update with response R is a convex combination of the current state and $R(x)$. The function $R(x) - x$ is continuous and strictly decreasing, with opposite signs at the endpoints. This proves invariance and equilibrium uniqueness for the stated response class. ■

Thus the inventory capacity bound remains valid even when an equilibrium becomes unstable. The derivative of the update supplies a separate stability criterion.

Theorem 1 (global equilibrium behavior). Fix $0 < \delta \le 1$. The centered case corresponds to $\theta = 1/2$. Define

$$\kappa_c = \frac{4(2-\delta)}{\delta}. \tag{7}$$

Part (a) applies to every admissible target position. Parts (b) and (c) concern the centered policy.

a) If $0 < \kappa < \kappa_c$, the unique equilibrium (equal to $x^* = \frac{1}{2}$ under the centered policy) is globally exponentially attractive on $[0,1]$.
b) If $\kappa = \kappa_c$, the equilibrium is globally attractive on $[0,1]$, is Lyapunov stable, and has multiplier -1.
c) If $\kappa > \kappa_c$, the equilibrium is linearly unstable.

Proof. Differentiate (5):

$$G'(y) = 1 - \delta - \frac{\delta\kappa}{4}\operatorname{sech}^2\left(\frac{\kappa y}{2}\right). \tag{8}$$

Since $0 < \operatorname{sech}^2(z) \le 1$, for all $y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$,

$$1 - \delta - \frac{\delta\kappa}{4} \le G'(\mathrm{y}) < 1. \tag{9}$$

For $\kappa < \kappa_c$, the lower bound exceeds -1, so $\max_{\left[-\frac{1}{2},\frac{1}{2}\right]} |G'| = L < 1$. The mean-value theorem gives $|G^n(y_0)| \le L^n |y_0|$, proving (a) for the centered policy.

For an arbitrary target position, the derivative of the sigmoid target has magnitude at most one quarter of the sensitivity. Thus differentiation of (3) yields the same uniform bounds as (9) for every target position. When the sensitivity is below the stated threshold, these bounds give a contraction on the capacity interval. This proves part (a) in full generality.

At $\kappa = \kappa_c$ the derivative at zero equals -1. The inequality $G'(y) > -1$ holds whenever $y \neq 0$. For $y > 0$, integrating $G'(t) > -1$ over $(0, y)$ gives $G(y) > -y$. Also $G(y) < y$, since $1 - \delta < 1$ and $\tanh\left(\frac{\kappa y}{2}\right) > 0$. Thus $|G(y)| < y$ for $y > 0$. Oddness extends this inequality to all nonzero y. The sequence $|y_n|$ decreases strictly until, if ever, it reaches zero. Its limit must be zero by continuity and the strict inequality. The same bound $|G(y)| \le |y|$ establishes Lyapunov stability, proving (b).

Finally, the equilibrium multiplier is

$$\lambda_* = G'(0) = 1 - \delta - \frac{\delta\kappa}{4}. \tag{10}$$

For $\kappa > \kappa_c$ it is less than -1, so the fixed point is unstable.■

For a non-centered policy $\theta \neq \frac{1}{2}$, Theorem 1(a) ensures global exponential convergence when $\kappa < \kappa_c$. At the unique equilibrium x^*, the local multiplier is $1 - \delta - \delta\kappa x^*(1 - x^*)$. Its instability threshold may depend on the equilibrium position. Section 4 uses the symmetry of the centered policy.

At the critical sensitivity, the equilibrium multiplier is –1. The derivative test leaves the convergence rate unresolved; the cubic term provides the leading-order behavior.

Theorem 2 (critical convergence rate). Suppose $\theta = \frac{1}{2}$, $\kappa = \kappa_c$, and $x_0 \in [0,1]$. If the orbit does not reach $x^* = \frac{1}{2}$ in finitely many steps, then

$$\lim_{n\to\infty} \sqrt{n} \left| x_n - \frac{1}{2} \right| = \sqrt{\frac{24}{\delta\kappa_c^3}}. \tag{11}$$

Proof. With $\beta = \frac{\delta\kappa_c^3}{48} > 0$, the Taylor expansion of (5) at the critical parameter is

$$G(y) = -y + \beta y^3 + O(y^5). \tag{12}$$

Theorem 1 gives $y_n \to 0$. For all sufficiently large n with $y_n \neq 0$, the sign of y_{n+1} is opposite to that of y_n. Writing $r_n = |y_n|$ then yields

$$r_{n+1} = r_n - \beta r_n^3 + O(r_n^5), \qquad r_n > 0. \tag{13}$$

For $z_n = r_n^{-2}$, the expansion $(1 - \varepsilon)^{-2} = 1 + 2\varepsilon + O(\varepsilon^2)$ with $\varepsilon = \beta r_n^2 + O(r_n^4)$ gives

$$z_{n+1} - z_n = 2\beta + O(r_n^2) \to 2\beta. \tag{14}$$

Averaging the increments shows $\frac{z_n}{n} \to 2\beta$, equivalently $n r_n^2 \to (2\beta)^{-1}$. Substituting β proves (11).■

The rate in (11) is shared by trajectories that never reach equilibrium exactly. Those reaching it in finitely many steps remain there, so their deviations have no positive asymptotic coefficient.

4 THE POST-THRESHOLD PERIOD-TWO BRANCH

Odd symmetry turns the search for a two-cycle into a scalar amplitude problem. We establish the branch and its stability before examining parameter dependence and global attraction.

Theorem 3 (uniqueness and stability of the period-two orbit). Let $0 < \delta \leq 1$, $\theta = \frac{1}{2}$ and $\kappa > \kappa_c$. Then (3) possesses exactly one orbit of prime period two, namely the symmetric orbit

$$\mathcal{O}_2: \frac{1}{2} - a \leftrightarrow \frac{1}{2} + a, \qquad 0 < a < \frac{1}{2}. \tag{15}$$

Its half-amplitude a is the unique positive solution of

$$\tanh\left(\frac{\kappa a}{2}\right) = Ba, \qquad B = \frac{2(2-\delta)}{\delta}. \tag{16}$$

This orbit is linearly asymptotically stable for every $\kappa > \kappa_c$.

Proof. As G is odd, a symmetric two-cycle requires $G(a) = -a$, giving (16). Define $H(a) = \frac{\tanh\left(\frac{\kappa a}{2}\right)}{a}$ for $a > 0$. Strict concavity of tanh on the positive half-line shows that H decreases strictly. As a decreases to zero, $H(a)$ tends to $\frac{\kappa}{2} > B$, because $\kappa > \kappa_c = 2B$. At the other endpoint, $H\left(\frac{1}{B}\right) = B\tanh\left(\frac{\kappa}{2B}\right) < B$. Hence $H(a) = B$ has exactly one solution in $\left(0, \frac{1}{B}\right)$; since $B \geq 2$, this root lies in $\left(0, \frac{1}{2}\right)$.

Let $u = \frac{\kappa a}{2} > 0$. Equation (16) yields $\tanh u = Ba$ and $\frac{\kappa}{\kappa_c} = \frac{u}{\tanh u}$. Consequently,

$$G'(a) = 1 - \delta - (2-\delta)\frac{2u}{\sinh(2u)}. \tag{17}$$

The inequality $0 < \frac{2u}{\sinh(2u)} < 1$ for $u > 0$ implies $-1 < G'(a) < 1 - \delta < 1$. Oddness of G makes its derivative even, so $G'(-a) = G'(a)$. The multiplier of the two-step return map is

$$\mu_2 = G'(a)G'(-a) = \left(G'(a)\right)^2, \qquad 0 \leq \mu_2 < 1. \tag{18}$$

To establish uniqueness among all prime period-two orbits, let $s \neq t$ satisfy $G(s) = t$ and $G(t) = s$. Adding these equations gives

$$\delta(s+t) = -\frac{\delta}{2}\left[\tanh\left(\frac{\kappa s}{2}\right) + \tanh\left(\frac{\kappa t}{2}\right)\right].$$

Since tanh is odd and strictly increasing, $\tanh\left(\frac{\kappa s}{2}\right) + \tanh\left(\frac{\kappa t}{2}\right)$ has the same strict sign as $s + t$ whenever $s + t \neq 0$. The equality is then impossible unless $s + t = 0$. Every nontrivial two-cycle is therefore symmetric, and the unique positive root of (16) identifies the only prime period-two orbit. Thus $\mathcal{O}_2$ is linearly asymptotically stable. ■

Theorem 3 establishes uniqueness and linear attraction of the period-two orbit. For $0 < \delta < 1$, the possible presence of higher-period orbits and the global basin structure require separate analysis.

The next theorem quantifies the separation between the two periodic stock levels, including its growth near onset and its limiting size.

Theorem 4 (monotonicity, onset scaling, and amplitude saturation). For fixed $0 < \delta \leq 1$, the half-amplitude $a(\kappa)$ in Theorem 3 is strictly increasing on (κ_c, ∞) and obeys

$$0 < a(\kappa) < \frac{\delta}{2(2-\delta)}. \tag{19}$$

Moreover,

$$a(\kappa) \sim \sqrt{\frac{12(\kappa-\kappa_c)}{\kappa_c^3}} \quad (\kappa \downarrow \kappa_c), \qquad \lim_{\kappa\to\infty} a(\kappa) = \frac{\delta}{2(2-\delta)}. \tag{20}$$

For a fixed sensitivity $\kappa > 4$, the half-amplitude also increases strictly with the adjustment fraction δ on the period-two regime $\delta > \frac{8}{(\kappa+4)}$.

Proof. Equation (16) can be inverted on $0 < a < \frac{1}{B}$:

$$\kappa = K(a) := \frac{2}{a}\operatorname{arctanh}(Ba). \tag{21}$$

Put $t = Ba \in (0,1)$. Then

$$K'(a) = \frac{2}{a^2}\left(\frac{t}{1-t^2} - \operatorname{arctanh}(t)\right) > 0, \tag{22}$$

The expression in braces in (22) vanishes at $t = 0$ and has derivative $\frac{2t^2}{(1-t^2)^2} > 0$ for $t > 0$. Hence K is strictly increasing. By (21), $K(a)$ tends to $2B = \kappa_c$ as a decreases to zero and diverges to infinity as a increases to $\frac{1}{B}$. Therefore, it has a strictly increasing inverse $a(\kappa)$. Expansion of $\operatorname{arctanh}(Ba) = Ba + \frac{B^3a^3}{3} + O(a^5)$ in (21) gives

$$\kappa - \kappa_c = \left(\frac{2B^3}{3}\right)a^2 + O(a^4).$$

Since $B = \frac{\kappa_c}{2}$, the onset scaling in (20) follows. For fixed $\kappa > 4$, the cycle exists for $\delta > \frac{8}{(\kappa+4)}$. The quantity $B = \frac{4}{\delta} - 2$ decreases strictly as δ increases. Since the function $H(a) = \frac{\tanh\left(\frac{\kappa a}{2}\right)}{a}$ is strictly decreasing in a by Theorem 3, equation (16) then forces the unique solution a to increase with the adjustment fraction. ■

The two stocks in the cycle differ by $2a$; hence their peak-to-trough separation is strictly less than $\frac{\delta}{(2-\delta)}$ of normalized capacity. The expansion in (20), together with the multiplier crossing at -1, identifies a supercritical flip bifurcation [13]. The adjustment-fraction dependence established in Theorem 4 describes how stronger stock correction enlarges the alternating separation.

For full adjustment, every nonzero centered deviation changes sign at each iteration. This permits a monotone amplitude argument and yields a global attraction result for the period-two orbit.

Proposition 2 (global attraction of the two-cycle for $\delta = 1$). Let $\delta = 1$, $\theta = \frac{1}{2}$, and $\kappa > 4$. Every initial condition with $0 \le x_0 \le 1$ other than $x_0 = \frac{1}{2}$ has distance tending to zero from the unique period-two orbit $\mathcal{O}_2$. Its two even- and odd-time subsequences approach the opposite points of that orbit.

Proof. For $\delta = 1$ the centered map becomes $G(y) = -h(y)$, where $h(y) = \frac{1}{2}\tanh\left(\frac{\kappa y}{2}\right)$ for $y \ge 0$. By oddness, G reverses sign and the positive amplitudes $r_n = |y_n|$ satisfy $r_{n+1} = h(r_n)$. The ratio $\frac{h(r)}{r}$ is strictly decreasing for $r > 0$, equals $\frac{\kappa}{4} > 1$ in the limit $r \downarrow 0$, and is less than one at $r = \frac{1}{2}$. Hence it has the unique positive fixed amplitude a from (16). For $0 < r < a$ one has $r < h(r) < a$, whereas for $a < r \le \frac{1}{2}$ one has $a < h(r) < r$. Therefore r_n converges monotonically to a from either side. Since the sign alternates at each step, the even and odd subsequences converge to the two members of $\mathcal{O}_2$. ■

For $0 < \delta < 1$, the positive linear component of G can interrupt the sign reversal used in the proof. The global attraction conclusion of Proposition 2 is therefore restricted to full adjustment.

5 NUMERICAL ILLUSTRATIONS AND DISCUSSION

The flip threshold derived in Section 3 provides a reference for interpreting the observed stock dynamics. The computations below compare numerical iteration with the analytical period-two branch and the critical convergence law.

For the centered policy with $\delta = 0.8$, the equilibrium multiplier reaches -1 at the flip threshold $\kappa c = 6$. Figure 1(a) shows the long-run stock levels over $3 \leq \kappa \leq 10$. A single attracting stock level gives way to two alternating levels above this threshold, consistent with Theorem 3. The three trajectories in Figures 1(b)–(d) start from $x_0 = 0.16$ and use $\kappa = 4, 6$, and 8, respectively. They show equilibrium convergence, slowly decaying critical oscillations, and approach to the stable period-two orbit.

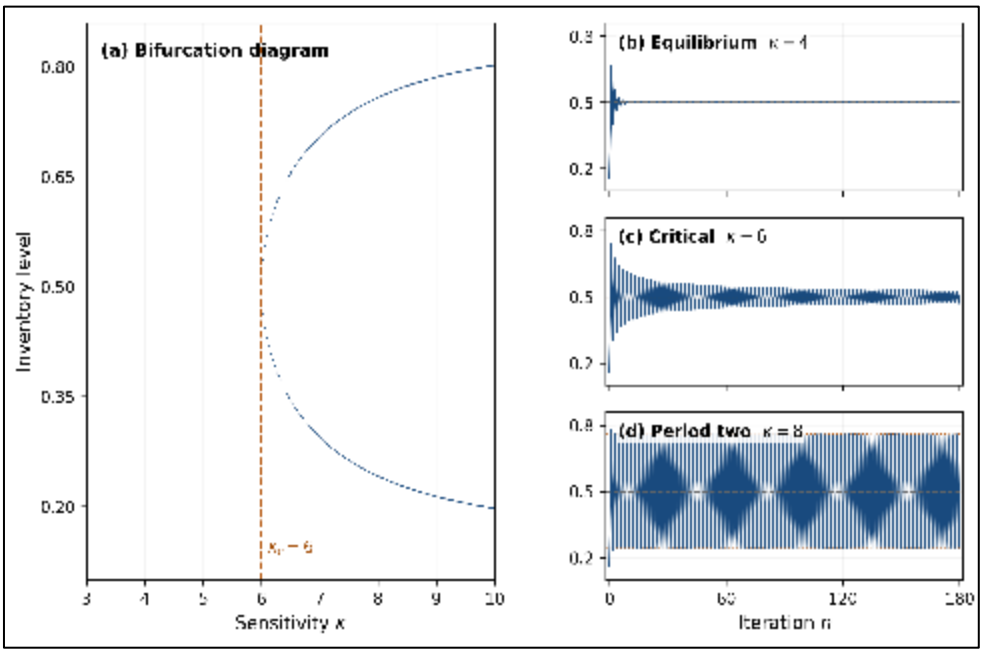


Figure 1: Flip transition and stock trajectories for the centered policy (δ = 0.8). (a) Numerical bifurcation diagram for 3 ≤ κ ≤ 10 after 12,000 transient iterations; 90 subsequent iterates are displayed at each sensitivity. The vertical dashed line marks $\kappa_c = 6$. (b)–(d) Stock trajectories at $\kappa = 4, 6$, and 8 from $x_0 = 0.16$ over 180 iterations. Horizontal dashed lines mark the equilibrium $x^* = 0.5$; dotted lines in (d) indicate the two-cycle stock levels.

Table 1 reports the two stock levels computed from the positive root of (16), together with the two-step multiplier. We solve the scalar equation by bracketing within $\left(0, \frac{1}{B}\right)$ and evaluate the cycle multiplier directly. The reported multipliers satisfy the strict stability bound established in Theorem 3.

Table 1: Analytic branch values evaluated numerically at $\delta = 0.8$.

κ	Lower stock $\frac{1}{2} - a$	Upper stock $\frac{1}{2} + a$	Two-step multiplier μ_2	Orbit residual
6.6	0.332353	0.667647	0.617964	$\leq 2 \times 10^{-16}$
8.0	0.241495	0.758505	0.191598	$\leq 2 \times 10^{-16}$
10.0	0.197556	0.802444	0.023561	$\leq 2 \times 10^{-16}$
16.0	0.170047	0.829953	0.018341	$\leq 2 \times 10^{-16}$

Figure 2 evaluates the period-two branches directly from the scalar root equation (16). This analytical construction agrees with the long-run orbits in Figure 1(a). The flip threshold moves to larger κ as δ decreases, as predicted by (7), and the amplitude ordering follows the monotonicity established in Theorem 4.

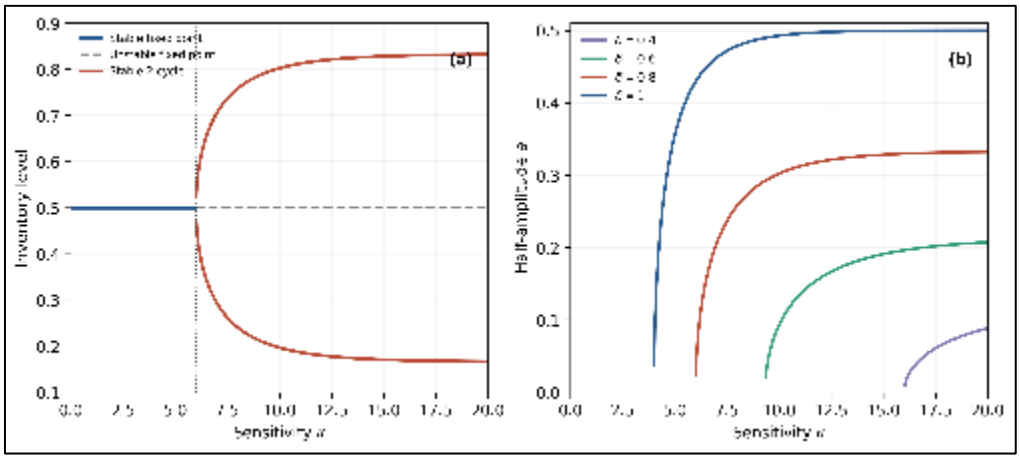


Figure 2: Analytical branch evaluation as replenishment sensitivity varies. (a) Fixed-point and unique period-two branches for $\delta = 0.8$; the solid equilibrium branch is stable, the dashed equilibrium branch is unstable, and the two red branches are the attracting period-two stocks. The vertical dotted line marks $\kappa_c = 6$. (b) Half-amplitude $a(\kappa)$ for $\delta = 0.4, 0.6, 0.8$ and 1.0. Curves are computed from (16), and the absence of a curve below each threshold means that no nontrivial period-two orbit exists there.

For $\delta = 0.8$ and $\kappa = 6$, Theorem 2 predicts the leading coefficient

$$C = \sqrt{\frac{24}{0.8 \times 6^3}} \approx 0.372678, \qquad \left|x_n - \frac{1}{2}\right| \sim Cn^{-\frac{1}{2}}. \tag{23}$$

The critical-rate diagnostic uses 60,000 iterations at the threshold. The rescaled deviation $\sqrt{n}\left|x_n - \frac{1}{2}\right|$ is approximately 0.372612 at $n = 10{,}000$ and 0.372663 at $n = 50{,}000$, approaching the asymptotic coefficient $C \approx 0.372678$. Figure 3 compares the observed deviation with the leading-order prediction across the iteration range. Agreement becomes closer as n increases.

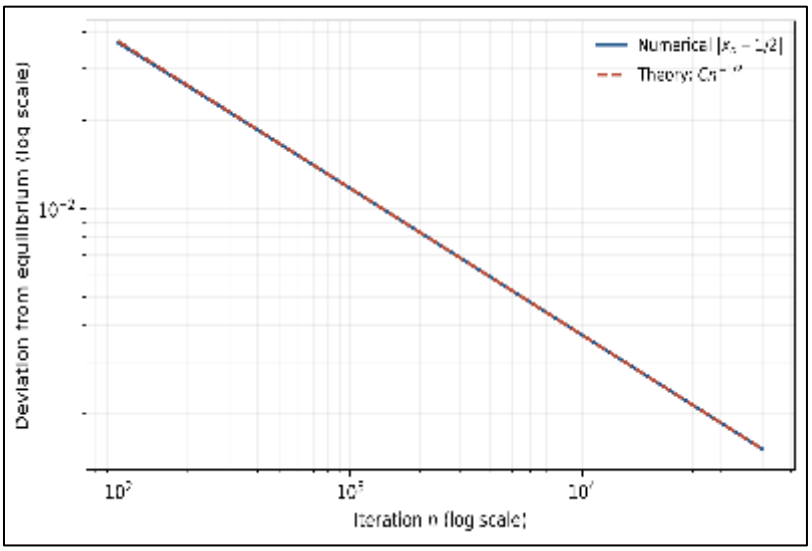


Figure 3: Critical convergence on logarithmic axes for $\delta = 0.8$, $\kappa = \kappa_c = 6$ and $x_0 = 0.16$. The solid curve represents the computed absolute deviation $\left|x_n - \frac{1}{2}\right|$ from 100 through 60,000 iterations, and the dashed curve is $Cn^{-1/2}$ with $C = \sqrt{\frac{24}{(\delta\kappa_c^3)}} \approx 0.372678$. The predicted slope is $-\frac{1}{2}$ (Theorem 2).

Capacity feasibility does not guarantee convergence to one stock level. Above the flip threshold, the locally attracting two-cycle stays inside the normalized interval, with separation determined by (16). The resulting fluctuations concern stock; bullwhip amplification is defined through order variability across a supply chain [8].

Increasing κ steepens the replenishment response near the target stock. Below κ_c, the centered equilibrium attracts trajectories exponentially; at the threshold, deviations decay algebraically; above it, the equilibrium is linearly unstable. The period-two amplitude grows toward the finite limit in (19). Increasing the adjustment fraction δ lowers the flip threshold and increases the cycle amplitude for fixed sensitivity within the period-two regime. The finite amplitude bound also limits the additional stock separation produced by arbitrarily large sensitivity. The normalized capacity sets this bound even though the equilibrium multiplier continues to decrease as sensitivity increases.

Uniqueness of the two-cycle follows from symmetry about $\frac{1}{2}$. The complete basin result proved here assumes $\delta = 1$. For smaller adjustment fractions, the global basin remains unresolved. Delays, stochastic demand or pipeline inventory could change the bifurcation scenario [6, 7, 10].

6 CONCLUSION

The capacity-bounded stock map admits a unique equilibrium for every target position, and the global contraction bound extends to non-centered policies. For a centered target, an explicit sensitivity threshold separates exponential convergence from linear equilibrium instability. At criticality, convergence follows an algebraic law with a common leading coefficient for trajectories that avoid the equilibrium in finite time. Above the threshold, the model has exactly one prime period-two orbit, which is symmetric and linearly attracting. Its amplitude grows with sensitivity, has square-root onset scaling and approaches a finite limit. Full adjustment yields global attraction to that orbit, except at the equilibrium.

A capacity-preserving inventory rule can therefore sustain endogenous stock oscillations. The explicit formulas describe the consequences of feedback sensitivity within this one-state model. Delayed replenishment, stochastic demand and asymmetric target positions require further analysis.

STATEMENTS ON COMPLIANCE WITH ETHICAL STANDARDS

Disclosure of conflict of interest

All authors declare no conflicts of interest.

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