



(RESEARCH ARTICLE)



ANALYTICAL CARRIER-COLLECTION RETENTION AND DEGRADATION MARGIN IN A SERIES VERTICAL-JUNCTION SILICON SOLAR CELL

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ABSTRACT

This work develops an analytical theory of carrier-collection retention during diffusion-length degradation in a series vertical-junction silicon solar cell. Starting from the reduced collection law $C(h;f,b)$, degradation is parameterized by $\gamma=L/L_0$ under the reference assumption of lifetime-driven degradation, with constant diffusion coefficient D , thickness H , and interface velocities S_f and S_b . The induced trajectory is $(h,f,b)=(h_0/\gamma, \gamma f_0, \gamma b_0)$. An exact retention law is obtained. It is shown that $\mathcal{R}_C(1)=1$ and that $\mathcal{R}_C(\gamma)$ tends linearly to zero as $\gamma \rightarrow 0^+$, which guarantees the existence of a retention threshold for every $0 < \eta < 1$. The directional derivative satisfies $\gamma \frac{d}{d\gamma} \ln C_\gamma = -\mathcal{S}_h + \mathcal{S}_f + \mathcal{S}_b$. Its exact factorization reveals a boundary separating an M regime of locally monotonic loss from an NM regime of transient compensation. A reduction to two invariants, $a=h_0 f_0$ and $d=h_0 b_0$, reduces every trajectory to a one-dimensional function and shows that any possible compensation is necessarily transient. A materialized numerical campaign in the $b_0=1$ section, complemented by three representative trajectories, illustrates the strong dependence of the end-of-life margin on the initial regime. For $\eta=0.90$, the representative cases yield margins of approximately 10.89%, 56.34%, and 99.53%. A generic conversion to irradiation fluence is finally established from $L^{-2}=L_0^{-2}+k_l \varphi$. The results explicitly distinguish diffusion-length degradation, transient compensation of the reduced model, and improvement in overall photovoltaic performance, which is not claimed.

Keywords: Silicon solar cell; Series vertical junctions; Diffusion length; Carrier collection; Degradation margin; Interface recombination.

1 INTRODUCTION

Minority-carrier collection in a silicon solar cell depends jointly on geometry, diffusive transport, lifetime, and interface recombination. The diffusion length $L=\sqrt{D\tau}$ condenses an essential part of this competition: it measures the spatial scale over which a minority carrier can diffuse before recombination. In devices exposed to radiation environments, lattice defects can reduce lifetime and diffusion length, constituting a classical solar-cell degradation mechanism. Radiation-damage literature commonly expresses this evolution through a relation of the form $L^{-2}=L_0^{-2}+k_l \varphi$, where φ is the fluence and k_l is a damage coefficient depending on the material and irradiation conditions.

Vertical-junction architectures have specifically been studied for their ability to shorten collection distances and sustain collection when the diffusion length becomes limited. Within the family of series vertical-junction silicon solar cells, previous work has examined, among other topics, optimum thickness under irradiation and the influence of transport and recombination parameters. Dia et al. treated an optimum thickness under irradiation in this architecture, using a fluence-dependent diffusion length. Consequently, the effect of irradiation on L and on a thickness criterion is not, by itself, sufficient novelty for the present work [2].

The question addressed here is different: for a fixed geometry and a given initial state, what fraction of diffusion-length degradation can be tolerated before collection falls below a prescribed fraction of its initial value? This formulation leads to a retention-based end-of-life quantity rather than to a new optimum thickness. It also differs from the effective thickness H_η defined relative to asymptotic collection: in the present problem, H is fixed and the transport state evolves.

An additional difficulty arises because a variation in L simultaneously changes the three reduced parameters $h=H/L$, $f=S_f L/D$, and $b=S_b L/D$. The degradation trajectory is therefore not an isolated variation of h . It can place the effects of reduced thickness, junction collection, and rear-interface recombination in competition. Within the collection law considered here, the present article exploits this structure to establish an exact retention law, an analytical boundary between monotonic degradation and transient compensation, and then an end-of-life degradation margin.

The objectives are therefore to: (i) derive the reduced trajectory associated with degradation of L ; (ii) establish the limits and existence of retention thresholds; (iii) factorize the derivative sign along the trajectory; (iv) identify the M and NM classes; (v) quantify the margin $\Delta L^\wedge(\eta)$; and (vi) relate this margin to a fluence when the radiation-damage law is applicable. The scope is deliberately theoretical: neither experimental validation nor a universal radiation-damage law is claimed.

2 PHYSICAL MODEL AND COLLECTION LAW

An $n^+/p/p^+$ series vertical-junction silicon solar cell is considered. Transverse transport of minority electrons in the p -type base is described, under steady-state conditions and within the one-dimensional diffusion approximation, by a continuity equation with constant coefficients. The adopted formulation provides the analytical basis used throughout the present study.

$$D \frac{d^2 \delta}{dx^2} - \frac{\delta}{\tau} + G = 0 \quad (1)$$

$$L = \sqrt{D\tau} \quad (2)$$

The interfaces are represented by recombination velocities S_f and S_b . With the reduced variables

$$h = \frac{H}{L}, \quad f = \frac{S_f L}{D}, \quad b = \frac{S_b L}{D} \quad (3)$$

the normalized collection response is

$$C(h; f, b) = \frac{f[\sinh + b(\cosh - 1)]}{(1 + bf)\sinh + (f + b)\cosh} \quad (4)$$

This expression is the canonical collection law of the diffusion model. In the reference model, $S_h > 0$, $S_f > 0$, and $S_b < 0$; it is also established that $S_h > |S_b|$ for $h, f, b > 0$.

3 DEGRADATION TRAJECTORY AND EXACT RETENTION LAW

3.1 Reference degradation assumption

We define $\gamma = L/L_0$, with $0 < \gamma \leq 1$. The case $\gamma = 1$ represents the initial state. The reference model assumes that the decrease in L results from a decrease in τ while D , H , S_f , and S_b remain constant. This assumption must be distinguished from more general degradation in which D or the interface velocities would also evolve.

$$h(\gamma) = \frac{h_0}{\gamma}, \quad f(\gamma) = \gamma f_0, \quad b(\gamma) = \gamma b_0 \quad (5)$$

The trajectory has two useful invariants:

$$hf = h_0 f_0 \equiv a, \quad hb = h_0 b_0 \equiv d(6)$$

3.2 Exact law

$$C_\gamma = \frac{\gamma f_0 \{\sinh(h_0/\gamma) + \gamma b_0 [\cosh(h_0/\gamma) - 1]\}}{[1 + \gamma^2 b_0 f_0] \sinh(h_0/\gamma) + \gamma(f_0 + b_0) \cosh(h_0/\gamma)}(7)$$

$$\mathcal{R}_c(\gamma) = \frac{C_\gamma}{c(h_0; f_0, b_0)}(8)$$

A compact form, convenient both analytically and numerically, is obtained by dividing the numerator and denominator by $\sinh(h_0/\gamma)$:

$$C_\gamma = \frac{\gamma f_0 [1 + \gamma b_0 \tanh(h_0/(2\gamma))]}{1 + \gamma^2 b_0 f_0 + \gamma(f_0 + b_0) \coth(h_0/\gamma)}(9)$$

4 ANALYTICAL PROPERTIES OF RETENTION

4.1 Limits

At the initial state, normalization directly gives

$$\mathcal{R}_c(1) = 1(10)$$

As $\gamma \rightarrow 0^+$, $h_0/\gamma \rightarrow \infty$, $\tanh(h_0/2\gamma) \rightarrow 1$, and $\coth(h_0/\gamma) \rightarrow 1$. Hence

$$C_\gamma \sim \frac{\gamma f_0 (1 + \gamma b_0)}{(1 + \gamma f_0)(1 + \gamma b_0)} = \frac{\gamma f_0}{1 + \gamma f_0} \sim \gamma f_0(11)$$

$$\mathcal{R}_c(\gamma) \sim \frac{f_0}{c_0} \gamma \rightarrow 0(12)$$

The normalized collection therefore ultimately vanishes for every positive initial condition of the model.

4.2 Existence of a retention threshold

For every $0 < \eta < 1$, continuity of $\mathcal{R}_c$ on $(0, 1]$, together with $\mathcal{R}_c(1) = 1$ and $\lim_{\gamma \rightarrow 0^+} \mathcal{R}_c = 0$, guarantees by the intermediate value theorem the existence of at least one solution of

$$\mathcal{R}_c(\gamma_\eta) = \eta(13)$$

This existence result is global over $h_0, f_0, b_0 > 0$. However, uniqueness of all roots does not follow from this property because the trajectory may be nonmonotonic.

5 ANALYTICAL BOUNDARY BETWEEN M AND NM REGIMES

5.1 Directional derivative

The logarithmic sensitivities of the collection law are $\mathcal{S}_h = \partial \ln C / \partial \ln h$, $\mathcal{S}_f = \partial \ln C / \partial \ln f$, and $\mathcal{S}_b = \partial \ln C / \partial \ln b$. Along the degradation trajectory, $d \ln h / d \ln \gamma = -1$ and $d \ln f / d \ln \gamma = d \ln b / d \ln \gamma = 1$. Therefore, exactly

$$\gamma \frac{d \ln C_\gamma}{d \gamma} = -\mathcal{S}_h + \mathcal{S}_f + \mathcal{S}_b(14)$$

This combination differs from the sensitivity transition $\mathcal{S}_h = \mathcal{S}_f$ studied previously: the simultaneous variation of b adds a directional contribution specific to degradation. The present problem therefore follows a coupled trajectory in reduced-parameter space rather than an isolated variation of one parameter.

5.2 Exact factorization

Let $x=h_0/\gamma$, $F=\gamma f_0$, and $B=\gamma b_0$. The total derivative can be written as

$$\frac{dC_\gamma}{d\ln\gamma} = \frac{F[A(x,B)-FK(x,B)]}{[BF\sinh x + B\cosh x + F\cosh x + \sinh x]^2} \quad (15)$$

The denominator is strictly positive. The two numerator functions are

$$A(x,B) = \sinh^2 x + B[2\sinh x(\cosh x - 1) - x\cosh x] + B^2[\cosh^2 x - \cosh x - x\sinh x] \quad (16)$$

$$K(x,B) = x + B[x\sinh x + \cosh x - 1] + B^2x(\cosh x - 1) \quad (17)$$

For $x, B > 0$, $K(x,B) > 0$. The derivative sign is therefore entirely governed by $\Phi = A - FK$. When $A > 0$, the local boundary is

$$F_{\text{deg}}(x,B) = \frac{A(x,B)}{K(x,B)} \quad (18)$$

Regime M is defined by $dC_\gamma/d\gamma > 0$: as γ physically decreases, collection decreases. Regime NM corresponds to $dC_\gamma/d\gamma < 0$: a decrease in γ locally increases collection, a behavior referred to here as transient compensation. If $A \leq 0$, $\Phi < 0$ for every $F > 0$.

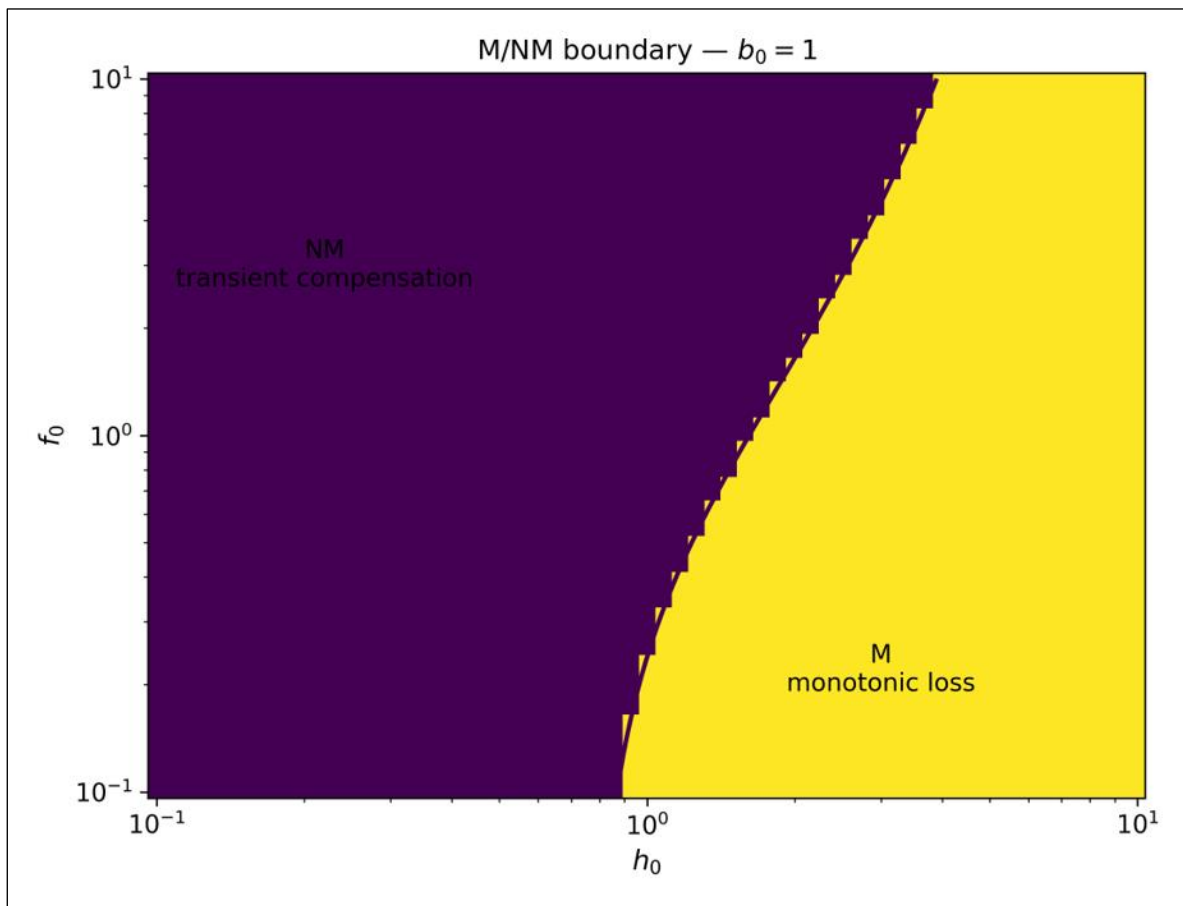


Figure 1: M/NM boundary in the (h_0, f_0) plane for $b_0=1$. The boundary corresponds to the change in the initial sign of the directional derivative

6 ONE-DIMENSIONAL REDUCTION AND ULTIMATE BEHAVIOR

The invariants in Eq. (6) allow the trajectory to be parameterized by $x=h=h_0/\gamma$:

$$f = \frac{a}{x}, \quad b = \frac{d}{x}, \quad x \geq h_0 \quad (19)$$

Collection along the trajectory then becomes

$$C_{\text{deg}}(x; a, d) = \frac{a[x+d \tanh(x/2)]}{x^2 + ad + (a+d)x \coth x} \quad (20)$$

As $x \rightarrow \infty$, $\tanh(x/2) \rightarrow 1$ and $\coth x \rightarrow 1$, so that

$$C_{\text{deg}}(x) \sim \frac{a(x+d)}{(x+a)(x+d)} = \frac{a}{x+a} \quad (21)$$

$$\frac{dC_{\text{deg}}}{dx} \sim -\frac{a}{(x+a)^2} < 0 \quad (22)$$

Thus, even when compensation appears at an early stage, it cannot persist to extreme degradation. Every trajectory ultimately enters a regime in which a further decrease in L reduces collection before C tends to zero. If the initial state is NM, continuity therefore implies the existence of at least one interior maximum. A global analytical proof of uniqueness of this extremum is not claimed.

7 DEFINITION OF THE END-OF-LIFE MARGIN

To avoid assuming a monotonicity that is not general, the EOL threshold is defined as the first crossing encountered as degradation progresses from $\gamma=1$. Mathematically, for $0 < \eta < 1$:

$$\gamma_{\eta}^{\text{EOL}} = \sup\{\gamma \in (0,1): \mathcal{R}_C(\gamma) = \eta\} \quad (23)$$

$$\Delta_L^{(\eta)} = 1 - \gamma_{\eta}^{\text{EOL}} \quad (24)$$

This definition remains valid for both M and NM classes. In the globally monotonic M class, strict monotonicity directly implies uniqueness of γ_{η} . In the NM class, the definition unambiguously selects the first EOL crossing even when uniqueness of all solutions is not assumed.

8 NUMERICAL RESULTS

8.1 Protocol

The retained numerical validation is deterministic. The materialized mapping uses a 61×61 logarithmic grid in the (h_0, f_0) plane, for $b_0=1$ and $0.1 \leq h_0, f_0 \leq 10$. At each point, the exact trajectory is evaluated to determine the first crossing $\mathcal{R}_C=0.90$ and the margin $\Delta_L^{(0.90)}$. This section constitutes an exploration domain of the model rather than a statistical distribution of real cells. It is complemented by three representative trajectories selected to illustrate a fragile M regime, an intermediate NM regime, and a strongly compensating NM regime.

The materialized campaign numerically confirms, within the section and for the representative trajectories investigated, the existence of monotonic behaviors and behaviors exhibiting transient compensation. It is not used here to claim an exhaustive classification of the three-dimensional (h_0, f_0, b_0) space, nor a global numerical proof of uniqueness of extrema or crossings. These properties remain open beyond the effectively materialized domain.

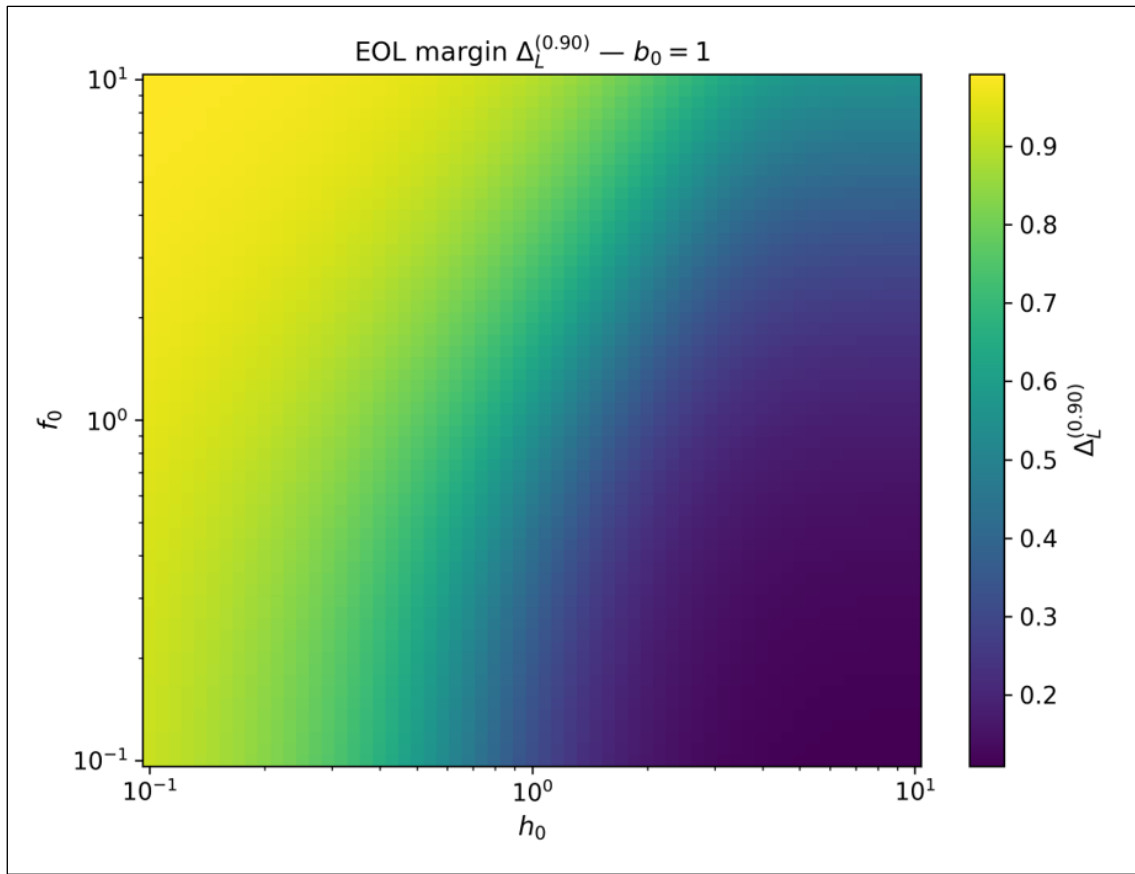


Figure 2: Map of the degradation margin $\Delta_L^{(0.90)}$ in the (h_0, f_0) plane for $b_0=1$.

8.2 Magnitude of the EOL margin

The three representative trajectories reveal a very wide range of EOL margins. The regime $(h_0, f_0, b_0) = (10, 0.1, 0.1)$ reaches the 90% threshold at $\gamma \approx 0.89108$, giving $\Delta_L^{(0.90)} \approx 10.89\%$. The case $(1, 1, 1)$ gives $\Delta_L^{(0.90)} \approx 56.34\%$, with a maximum retention of approximately 1.0377 at $\gamma \approx 0.7584$. Finally, the regime $(0.1, 10, 10)$ gives $\gamma \approx 0.004683$ and $\Delta_L^{(0.90)} \approx 99.53\%$, with strong transient compensation and a maximum of approximately 6.5978 at $\gamma \approx 0.07574$. These values characterize these representative cases and are not presented as global extrema of the entire parameter space.

These three regimes illustrate the diversity of behaviors produced by the exact law: monotonic loss, moderate compensation, and strong compensation. The transient increase in $\mathcal{R}_C$ in NM regimes does not constitute an improvement in overall photovoltaic efficiency.

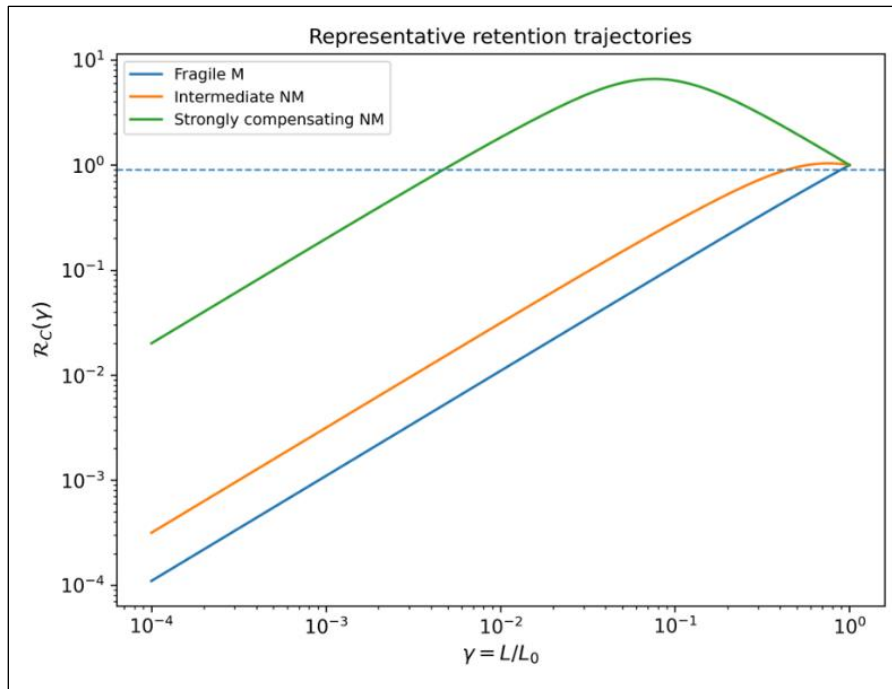


Figure 3: Representative retention trajectories: fragile M regime, intermediate NM regime, and strongly compensating NM regime. The horizontal line marks $\mathcal{R}_C=0.90$.

8.3 Compensation gain

The maximum compensation gain is defined by $\max_{\gamma} \mathcal{R}_C(\gamma)$. In an M regime, this quantity equals 1 because the initial state is the maximum. In an NM regime, it exceeds 1. This transient increase must not be interpreted as an improvement in overall photovoltaic efficiency: it concerns only the normalized collection response of the reduced model. It results from the competition between h , which increases, and f and b , which decrease as L degrades.

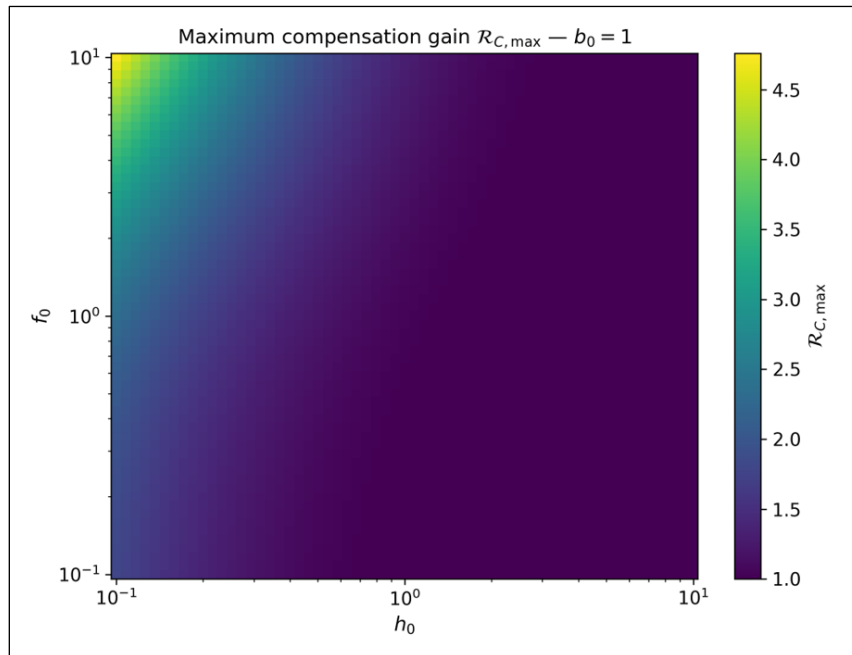


Figure 4: Maximum compensation gain in the (h_0, f_0) plane for $b_0=1$.

9 APPLICATION TO A RADIATION-DAMAGE LAW

Radiation degradation of solar cells is classically associated with decreases in lifetime and diffusion length. A commonly used form is $L^{-2} = L_0^{-2} + k_l \phi$. This relation is documented in radiation-damage models and is consistent with experimental studies reporting reductions in lifetime and diffusion length under irradiation [4,5].

$$\frac{1}{L^2} = \frac{1}{L_0^2} + k_l \phi(25)$$

With $\gamma=L/L_0$, one obtains

$$\gamma(\phi) = \frac{1}{\sqrt{1+k_l \phi L_0^2}}(26)$$

The EOL threshold therefore converts to

$$\phi_\eta^{\text{EOL}} = \frac{(\gamma_\eta^{\text{EOL}})^{-2} - 1}{k_l L_0^2}(27)$$

It is preferable to retain this relation in reduced form as long as no specific material-particle-energy combination and no experimentally documented k_l coefficient are imposed. The present work therefore does not assign an artificial universal absolute fluence.

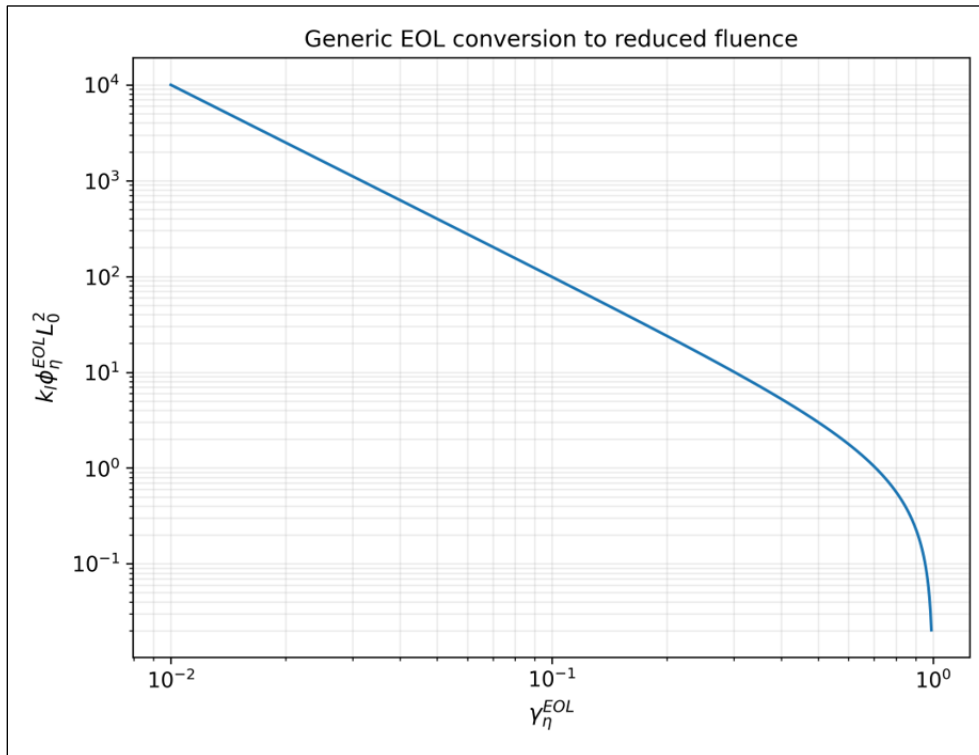


Figure 5: Generic conversion of the EOL threshold to the reduced fluence $k_l \phi_\eta^{\text{EOL}} L_0^2$.

10 DISCUSSION

The first contribution of this work is to replace a scalar view of L degradation with a coupled trajectory in (h, f, b) space. Even if H , D , S_f , and S_b remain constant, a variation in L causes $h \propto L^{-1}$ while $f, b \propto L$. This trajectory geometry explains why the collection response cannot be inferred from sensitivity to h alone.

The second contribution is the M/NM boundary. The relation $\gamma \ln C / d\gamma = -S_h + S_f + S_b$ shows that degradation explores a specific direction in parameter space. It therefore reproduces neither the transition $S_h = S_f$ of the sensitivity analysis nor the effective-thickness criterion $C(H_\eta) = \eta C_\infty$. The three quantities answer three distinct questions: local parameter dominance, the thickness required to reach a fraction of the asymptote, and the admissible degradation margin before losing a fraction of the initial collection.

The third contribution is the robust EOL definition. Nonmonotonicity prevents γ_η from being naively defined as an assumed unique root. Selecting the supremum identifies the first crossing encountered as degradation progresses from $\gamma=1$ and therefore remains operational in both classes. The materialized calculations illustrate this construction without constituting a global proof of uniqueness of extrema or of all crossings for $h_0, f_0, b_0 > 0$.

Transient compensation requires careful interpretation. A high b_0 corresponds to a high reduced rear-recombination parameter and, for given S_b , D , and L_0 , to a stronger rear-recombination penalty on collection. As L decreases in the reference model, $b = \gamma b_0$ also decreases, progressively reducing this penalty and potentially compensating other effects temporarily. This mathematical property does not mean that real irradiation improves the cell: irradiation can simultaneously modify τ , D , mobility, defect density, and surface recombination. Measurements on c-Si cells irradiated by low-energy electrons and protons indeed show decreases in lifetime and diffusion length, emphasizing the limited scope of the reference assumption [5].

A natural extension is therefore to introduce $d_D = D/D_0$ and $t = \tau/\tau_0$, with $\gamma = \sqrt{(d_D t)}$. If H, S_f, S_b remain fixed, then $h = h_0/\gamma$ while $f = f_0\gamma/d_D$ and $b = b_0\gamma/d_D$. The trajectory is then no longer determined by γ alone. This extension is left for future work in order to preserve an analytically interpretable result here.

Finally, previous literature on vertical junctions and irradiation has already addressed vertical architectures, diffusion-length limitations, and, for series vertical-junction cells, optimum thickness under irradiation [2,3]. The present work therefore claims neither the discovery of radiation-induced degradation of L nor the radiation tolerance of vertical architectures. The specific contribution examined here is narrower: for the collection law considered, it formulates a coupled degradation trajectory, a retention function $\mathcal{R}_C$, a directional M/NM boundary, and an EOL margin $\Delta_L(\eta)$. This novelty statement is limited to the audited bibliographic corpus and is not a claim of universal exhaustiveness.

11 CONCLUSION

An analytical theory of collection retention under diffusion-length degradation has been developed for a series vertical-junction silicon solar cell. Under the reference assumption of lifetime-driven degradation of L with constant D , H , S_f , and S_b , the reduced state follows $(h, f, b) = (h_0/\gamma, \gamma f_0, \gamma b_0)$. The exact retention law satisfies $\mathcal{R}_C(1) = 1$ and $\mathcal{R}_C \rightarrow 0$ as $\gamma \rightarrow 0^+$, guaranteeing the existence of every threshold $0 < \eta < 1$.

The derivative along the trajectory reduces to $-\mathcal{S}_h + \mathcal{S}_f + \mathcal{S}_b$ and admits an analytical factorization that separates an M regime of monotonic loss from an NM regime of transient compensation. The reduction through the invariants $h f$ and $h b$ further shows that any compensation is necessarily transient because collection ultimately tends to zero. A global proof of uniqueness of the extremum in the NM class is not claimed.

The materialized numerical calculations confirm the two essential behaviors predicted by the analysis: monotonic loss and transient compensation. For the three representative trajectories, the margin $\Delta_L(0.90)$ is approximately 10.89%, 56.34%, and 99.53%, highlighting the strong dependence of degradation tolerance on the nominal state. These numerical results are illustrative and are not extrapolated into a claim of global uniqueness. The conversion $\phi_{\eta}^{\text{EOL}} = [(\gamma_{\eta}^{\text{EOL}})^{-2} - 1]/(k_l L_0^2)$ finally relates this margin to a radiation-damage law when k_l is known.

The proposed framework therefore provides an end-of-life criterion distinct from optimum-thickness or effective-thickness criteria: it quantifies the amount of transport degradation that can be tolerated before a prescribed level of collection is lost, while identifying regimes in which transient compensation can temporarily mask degradation.

STATEMENTS ON COMPLIANCE WITH ETHICAL STANDARDS

Disclosure of conflict of interest

The authors declare that they have no conflicts of interest.

Data availability

The numerical results presented in this study are based on the analytical model and deterministic parameter exploration described in the manuscript.

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APPENDIX A. SUMMARY OF ESTABLISHED ANALYTICAL RESULTS

- Trajectory: $(h,f,b)=(h_0/\gamma,\gamma f_0,\gamma b_0)$, with $h f$ and $h b$ invariant.
- Limits: $\mathcal{R}_C(1)=1$ and $\mathcal{R}_C(\gamma)\sim(f_0/C_0)\gamma\rightarrow 0$ as $\gamma\rightarrow 0^+$.
- Existence: for every $\eta\in(0,1)$, at least one solution of $\mathcal{R}_C(\gamma)=\eta$ exists.
- Derivative: $\gamma \ln C/d\gamma=-\mathcal{S}_h+\mathcal{S}_f+\mathcal{S}_b$.
- Local boundary: $A(h,b)-fK(h,b)=0$, with $K>0$.
- Reduction: $C_{\text{deg}}(x;a,d)=a[x+d \tanh(x/2)]/[x^2+ad+(a+d)x \coth x]$.
- Ultimate behavior: $C_{\text{deg}}\sim a/(x+a)$, so any possible compensation is transient.
- Global uniqueness of extrema in the NM class: not analytically demonstrated and not globally claimed in the present work.