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Mathematical development of the new Sine type II generalized Topp-Leone exponential distribution with applications to biomedical data

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Abstract

The development of flexible probability distributions remains an essential aspect of modern statistical modeling, particularly in capturing complex patterns observed in real-life data. In this study, a new continuous distribution, referred to as the New Sine II Generalized Topp-Leone Exponential (NSTIIGTLE) distribution, is introduced. The proposed model is constructed by integrating the sine Generator with the Type II generalized Topp-Leone family, using exponential distribution as the baseline. Key statistical properties of the new distribution are derived, including explicit expressions for the pdf, cdf, survival function, and hazard rate function. In addition, important characteristics such as moments, moment generating function, and quantile function are obtained to facilitate practical implementation and interpretation. Parameter estimation is carried out using the maximum likelihood estimation (MLE) and the maximum product of spacing MPS) methods, and the performance of the estimators is examined through a simulation study under varying sample sizes. The results indicate that the estimators are consistent and exhibits improve accuracy as the sample size increases. To demonstrate the practical relevance, the proposed distribution is applied to biomedical data, where it is compared with several existing competing models using standard goodness-of-fit measures such as AIC, BIC, and LL. The findings reveal that the NSTIIGTLE distribution provides a superior fit, highlighting its potential as a reliable tool for modeling lifetime data and other biomedical phenomena.

Keywords: Parameters; Exponential; Likelihood; Distributions; Topp-Leone

1. Introduction

Statistical modeling plays a fundamental role in understanding uncertainty and variability I real-world phenomena. In many scientific fields such as medicine, engineering, finance and environmental science, probability distributions are used to describe the behavior of random variables and to analyze lifetime or survival data. The effectiveness of statistical inference and prediction often depends on the suitability and flexibility of the probability distribution chosen to represent the underlying data generating mechanism. Consequently, the development of a new probability distributions has become an important area of research in statistical theory and applications.

Classical probability distributions such as the exponential, Weibull and gamma models have been widely used in reliability theory and survival analysis. Among these, the exponential distribution is particularly notable due to its simple mathematical structure and the memoryless property, which makes it convenient for modelling time-to-failure phenomena. However, despite its analytical tractability, the exponential distribution is limited by its single parameter structure and constant hazard rate, which restrict its ability to adequately capture complex patterns observed in real data. These limitations have motivated statisticians to develop generalized and extended forms of classical distributions that can provide greater flexibility in modelling diverse datasets.

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In order to improve the modelling capability of classical distributions, researchers have developed several generator mechanisms that produce new families of distributions by adding shape parameters. For example, the beta-generated family of distributions proposed by Eugene Jones (2004) introduced a flexible framework for creating new probability models using the beta distribution as a generator. Similarly, the Kumaraswamy-generated family was introduced by Cordeiro Gauss and de Casrto, M. (2011) to provide an alternative method for constructing generalized distributions with tractable mathematical properties. Other important generalization approaches include the exponentiated-G family proposed by Saralees Nadarajah and Samuel Kotz (2006) and the Marshal-Olkin family introduced by Albert W. Marshall and Ingram Olkin (1997). These generator techniques have significantly expanded the range of probability models available for statistical analysis.

More recently, several researchers have focused on extending the Topp-Leone distribution to generate more flexible models for lifetime data. For instance, the Type II generalized Topp-Leone-G (TIIGTL-G) family was introduced by Rezaei S., et al. (2017), providing a general mechanism for generating new distributions by combining the Topp-Leone transformation with a baseline distribution. This family has attracted attention because it can produce probability density functions with a variety of shapes and hazard rate functions capable of describing different reliability behaviors.

Further developments in distribution theory have explored the use of trigonometric functions to generate flexible probability models. Trigonometric-based generators, particularly sine and cosine transformations, have been shown to produce additional shape flexibility for modeling skewed or heavy-tailed datasets. Building upon these development, Isa Abdullahi et al. (2023) proposed the Sine Type II Topp-Leone-G (STIITL-G) family of distributions. The family integrates the sine transformation with the Type II Topp-Leone generator to create a flexible class of continuous distributions.

Motivated by the need for more adaptable probability models, this study focuses on the development of a new distribution obtained by combining the Sine Type II generalized Topp-Leone generator with the exponential baseline distribution.

2. Methodology

The cdf and pdf of exponential distribution, which serves as our baseline distribution with parameter θ are given by:

$$H(x;q) = 1 - e^{-qx}, \quad x > 0, q > 0 \quad \text{.....................(1)}$$

and

$$h(x;q) = qe^{-qx}, \; x > 0, q > 0 \quad \text{.....................(2)}$$

The cdf and pdf of the New Sine Type II Generalized Topp-Leone (NSTIIGTL) distribution, is given below:

$$f(x;\alpha,\beta,\theta) = \frac{\pi}{2} 2\alpha\beta h(x;\theta)[H(x;\theta)]^{2\beta-1}\left[1 - H(x;\theta)^{2\beta}\right]^{\alpha-1} Cos\left[\frac{\pi}{2}\left[1 - \left[1 - H(x;\theta)^{2\beta}\right]^{\alpha}\right]\right] \text{.....................(3)}$$

$$F(x;\alpha,\beta,\theta) = Sin\left[\frac{\pi}{2}\left[1 - \left[1 - H(x;\theta)^{2\beta}\right]^{\alpha}\right]\right] \text{.....................(4)}$$

Where a, b are shape parameters and θ represents a vector parameter from any baseline distribution.

The New Sine Type II Generalized Topp-Leone (NSTIIGTLE) distribution has cdf and pdf as follows:

$$F(x;\ \alpha,\beta,\theta) = Sin\left[\frac{\pi}{2}\left[1 - \left[1 - \left[1 - e^{-\theta x}\right]^{2\beta}\right]^{\alpha}\right]\right],\ x > 0, \alpha > 0, \beta > 0, \theta > 0 \text{.....................(5)}$$

And

$$f(x;\ \alpha,\beta,\theta) = \frac{\pi}{2} 2\alpha\beta\theta e^{-\theta x}\left[1 - e^{-\theta x}\right]^{2\beta-1}\left[1 - \left[1 - e^{-\theta x}\right]^{2\beta}\right]^{\alpha-1} Cos\left[\frac{\pi}{2}\left[1 - \left[1 - \left[1 - e^{-\theta x}\right]^{2\beta}\right]^{\alpha}\right]\right] \text{.....................(6)}$$

where θ is a scale parameter and a, b are shape parameters.

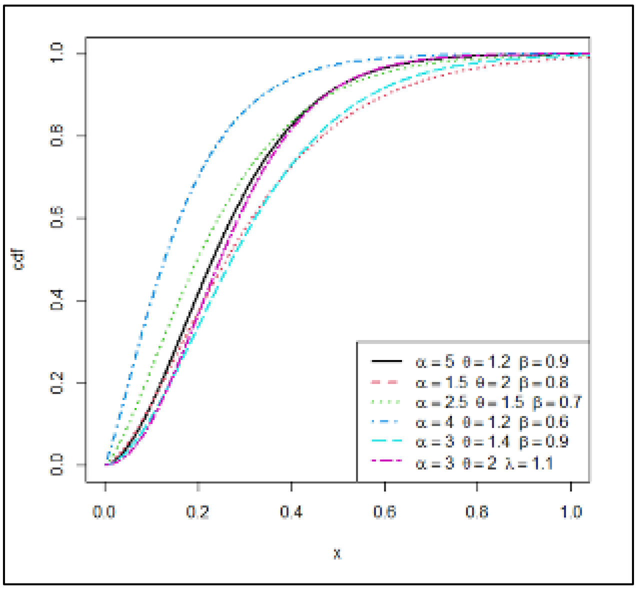

Figure 1 Plots of the cdf of NSTIIGTLE distribution with different parameter values

As seen from figure 1, we can deduced that the cdf has varying shapes. Some are steeper, while others are flatter and this indicates that the parameters significantly influence the overall shape of the distribution.

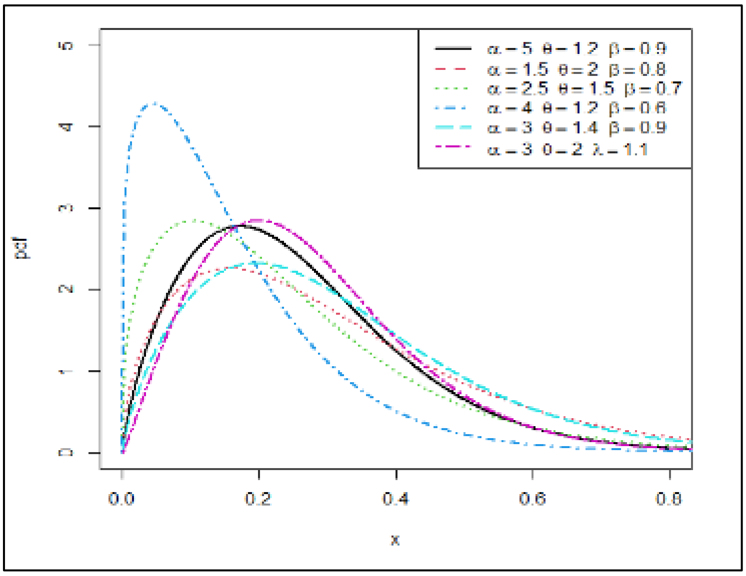

Figure 2 Plots of the pdf of NSTIIGTLE distribution under various parameter values are presented. The pdf plot shown in figure 2 explores how different parameter combination affect the shape of the distribution, and these parameters control the location, spread, and skewness of the distribution. We can see that the distribution has varying shapes. Some are more symmetric, while others are skewed. This indicates that the parameters have a substantial impact on the overall shape of the distribution

3. Validity Check on the New Sine Type II Generalized Topp-Leone Exponential (NSTIIGTLE) Distribution

It suffices to show that

$$\int_0^\infty f(x,\alpha,\beta,\theta)dx = 1 \quad \textbf{............(7)}$$

$$\int_0^\infty \frac{\pi}{2}2\alpha\beta\theta e^{-\theta x}\left[1-e^{-\theta x}\right]^{2\beta-1}\left[1-\left[1-e^{-\theta x}\right]^{2\beta}\right]^{\alpha-1} Cos\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x}\right]^{2\beta}\right]^{\alpha}\right]\right] \mathrm{dx}$$

$Let\ y = \frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x}\right]^{2\beta}\right]^{\alpha}\right]$, when $x \Longrightarrow 0, y \Longrightarrow 0$; when$x \Longrightarrow 0, y \Longrightarrow \frac{\pi}{2}$;

$$dx = \frac{dy}{\frac{\pi}{2}2\alpha\beta\theta e^{-\wp x}[1-e^{-\theta x}]^{2\beta-1}[1-[1-e^{-\theta x}]^{2\beta}]^{\alpha-1}}$$

Substituting into the original integral, we get

$$\int_0^{\frac{\pi}{2}} \frac{\pi}{2}2\alpha\beta\theta e^{-\wp x}\left[1-e^{-\theta x}\right]^{2\beta-1}\Big[1 - \left[1-e^{-\theta x}\right]^{2\beta}\Big]^{\alpha-1} Cos(y)\frac{dy}{\frac{\pi}{2}2\alpha\beta\theta e^{-\theta x}[1-e^{-\theta x}]^{2\beta-1}[1-[1-e^{-\theta x}]^{2\beta}]^{\alpha-1}}$$

$$\int_0^{\frac{\pi}{2}} Cos(y)dy = [Sin(y)]|_0^{\frac{\pi}{2}} = Sin\left(\frac{\pi}{2}\right) - Sin(0) = 1 - 0 = 1$$

Therefore,

$$\boldsymbol{f}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta}) = \frac{\pi}{2}2\alpha\beta\theta e^{-\theta x}\left[1-e^{-\theta x}\right]^{2\beta-1}\left[1-\left[1-e^{-\theta x}\right]^{2\beta}\right]^{\alpha-1} Cos\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x}\right]^{2\beta}\right]^{\alpha}\right]\right]$$

Is a valid pdf

4. Suitable Expansion of Density for the NSTIIGTLE Distribution

In this section, the expanded expressions of the pdf and cdf for the NSTIIGTLE distribution using standard mathematical techniques is derived. These include the Maclaurin series expansions for sine and cosine functions, along with the binomial expansion, expressed as follows:

$$Sin(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{..........................8}$$

$$Cos(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n!)} \quad \text{..................9}$$

$$(1-y)^b = \sum_{k=0}^{\infty}(-1)^k\binom{b}{k}y^k$$

The suitable expansion for the density function of the NSTIIGTLE distribution's pdf is derived through the application of Maclaurin series expansion for the cosine function and the binomial expansion, as outlined in equations (9) and (10), to equation (6).

$$\boldsymbol{f}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\theta})=\frac{\pi}{2}2\alpha\beta\theta e^{-\theta x}\left[1-e^{-\theta x}\right]^{2\beta-1}\left[1-\left[1-e^{-\theta x}\right]^{2\beta}\right]^{\alpha-1}Cos\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x}\right]^{2\beta}\right]^{\alpha}\right]\right]$$

$$Cos\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x}\right]^{2\beta}\right]^{\alpha}\right]\right]=\sum_{a=0}^{\infty}\frac{(-1)^{a}\pi^{2a}}{(2a)!\,2^{2a}}\left[1-\left[1-\left[1-e^{-\theta x}\right]^{2\beta}\right]^{\alpha}\right]^{2a}$$

$$\left[1-\left[1-e^{-\theta x}\right]^{2\beta}\right]^{\alpha(1+t)-1}=\sum_{p=0}^{\infty}(-1)^{p}\binom{\alpha(1+t)-1}{p}\left[1-e^{-\theta x}\right]^{2\beta p}$$

$$\left[1-e^{-\theta x}\right]^{2\beta(1+p)-1}=\sum_{l=0}^{\infty}(-1)^{l}\binom{2\beta(1+p)-1}{l}\left[e^{-\theta x}\right]^{l}$$

$$\boldsymbol{f}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta},\theta)=\sum_{\boldsymbol{a},\boldsymbol{t},\boldsymbol{p},l=0}^{\infty}\frac{(-1)^{a+t+p+l}\pi^{2a+1}2\alpha\beta\theta}{(2a)!\,2^{2a+1}}\binom{2a}{t}\binom{\alpha(1+t)-1}{p}\binom{2\beta(1+p)-1}{l}\left[e^{-\theta x}\right]^{l}$$

The probability density function (pdf) above can be rewritten, as follows:

$\boldsymbol{f}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta},\theta)=\sum_{\boldsymbol{a},\boldsymbol{t},\boldsymbol{p},l=0}^{\infty}\boldsymbol{\Phi}\left[e^{-\theta x}\right]^{l+1}$ ………………..(11)

where

$$\boldsymbol{\Phi}=\sum_{\boldsymbol{a},\boldsymbol{t},\boldsymbol{p},l=0}^{\infty}\frac{(-1)^{a+t+p+l}\pi^{2a+1}2\alpha\beta\theta}{(2a)!\,2^{2a+1}}\binom{2a}{t}\binom{\alpha(1+t)-1}{p}\binom{2\beta(1+p)-1}{l}$$

Also, the cumulative distribution function (cdf) can be expressed in expanded form as follows:

$$\boldsymbol{F}(\boldsymbol{x};\boldsymbol{\alpha},\boldsymbol{\beta},\theta)=\boldsymbol{Sin}\left[\frac{\boldsymbol{\pi}}{\boldsymbol{2}}\left[\boldsymbol{1}-\left[\boldsymbol{1}-\left[\boldsymbol{1}-\boldsymbol{e}^{\theta x}\right]^{\boldsymbol{2\beta}}\right]^{\boldsymbol{\alpha}}\right]\right]$$

$$\boldsymbol{Sin}\left[\frac{\boldsymbol{\pi}}{\boldsymbol{2}}\left[\boldsymbol{1}-\left[\boldsymbol{1}-\left[\boldsymbol{1}-\boldsymbol{e}^{\theta x}\right]^{\boldsymbol{2\beta}}\right]^{\boldsymbol{\alpha}}\right]\right]=\sum_{\boldsymbol{g=0}}^{\infty}\frac{(\boldsymbol{-1})^{\boldsymbol{g}}\boldsymbol{\pi}^{\boldsymbol{2g+1}}}{(\boldsymbol{2g+1})!\,\boldsymbol{2}^{\boldsymbol{2c+1}}}\left[\boldsymbol{1}-\left[\boldsymbol{1}-\left[\boldsymbol{1}-\boldsymbol{e}^{-\theta x}\right]^{\boldsymbol{2\beta}}\right]^{\boldsymbol{\alpha}}\right]^{\boldsymbol{2g+1}}$$

$$\left[\boldsymbol{1}-\left[\boldsymbol{1}-\left[\boldsymbol{1}-\boldsymbol{e}^{\theta x}\right]^{\boldsymbol{2\beta}}\right]^{\boldsymbol{\alpha}}\right]^{\boldsymbol{2g+1}}=\sum_{\boldsymbol{w=0}}^{\infty}(\boldsymbol{-1})^{\boldsymbol{w}}\binom{\boldsymbol{2g+1}}{\boldsymbol{w}}\left[\boldsymbol{1}-\left[\boldsymbol{1}-\boldsymbol{e}^{-\theta x}\right]^{\boldsymbol{2\beta}}\right]^{\boldsymbol{\alpha w}}$$

$$\left[\boldsymbol{1}-\left[\boldsymbol{1}-\boldsymbol{e}^{\theta x}\right]^{\boldsymbol{2\beta}}\right]^{\boldsymbol{\alpha w}}=\sum_{\boldsymbol{d=0}}^{\infty}(\boldsymbol{-1})^{\boldsymbol{d}}\binom{\boldsymbol{\alpha w}}{\boldsymbol{d}}\left[\boldsymbol{1}-\boldsymbol{e}^{-\theta x}\right]^{\boldsymbol{2\beta d}}$$

$$\left[\boldsymbol{1}-\boldsymbol{e}^{-\theta x}\right]^{\boldsymbol{2\beta d}}=\sum_{\boldsymbol{s=0}}^{\infty}(\boldsymbol{-1})^{\boldsymbol{s}}\binom{\boldsymbol{2\beta d}}{\boldsymbol{S}}\left[\boldsymbol{e}^{-\theta x}\right]^{\boldsymbol{s}}$$

$$F(x;\ \alpha,\beta,\theta)=\sum_{g,w,d,s=0}^{\infty}\frac{(-1)^{g+w+d+s}\pi^{2g+1}}{(2g+1)!\,2^{2c+1}}\binom{2g+1}{w}\binom{\alpha w}{d}\binom{2\beta d}{S}\left[\boldsymbol{e}^{-\theta x}\right]^{\boldsymbol{s}}$$

The expression of the cdf above can be rewritten as follows:

$F(x;\ \alpha,\beta,\theta)=\sum_{g,w,d,s=0}^{\infty}\Upsilon\left[\boldsymbol{e}^{-\theta x}\right]^{\boldsymbol{s}}$ ………………..(12)

Where, $\Upsilon = \frac{(-1)^{g+w+d+s}\pi^{2g+1}}{(2g+1)!2^{2c+1}}\binom{2g+1}{w}\binom{\alpha w}{d}\binom{2\beta d}{S}$

5. Statistical Properties of the NSTIIGTLE Distribution

In this section, various statistical properties of the NSTIIGTLE distribution were investigated.

5.1. Moments of the NSTIIGTLE Distribution

Given the significance of moments in statistical studies, particularly for real-world applications, we proceed to derive the r^{th} moments for NSTIIGTLE.

$$E(X^r) = \int_0^\infty x^r f(x)dx \text{ ………………..(13)}$$

The r^{th} moments of the NSTIIGTLE distribution are obtained by substituting equation (11) into equation (13), resulting in:

$$E(X^r) = \sum_{a,t,p,l=0}^{\infty} \Phi \int_0^\infty x^r \left[e^{-\theta x}\right]^{l+1} dx \text{ ………………..(14)}$$

Consider the integral part of equation (11)

$$\int_0^\infty x^r \left[e^{-\theta x}\right]^{l+1} dx$$

Let, $y \quad (l \quad 1)qx \quad x \quad \frac{y}{(l+1)q}; dx \quad \frac{dy}{(l+1)q}$

Then

$$\int_0^\infty \left[\frac{y}{(l+1)\theta}\right]^r e^{-y} \frac{dy}{(l+1)\theta}$$

$$\frac{1}{[(l+1)\theta]^{r+1}} \int_0^\infty y^r e^{-y} dy$$

$$\int_0 y^r e^{-y} dy = \mathrm{G}(r+1)$$

So therefore

$$E(X^r) = \sum_{a,t,p,l=0} \frac{\mathrm{F}\,\mathrm{G}(r+1)}{(l+1)^{r+1} q^r} \text{ ………………..(15)}$$

now

$$\mathbf{\Phi} = \frac{(-1)^{a+t+p+l}\pi^{2a+1}2\alpha\beta\theta}{(2a)!\,2^{2a+1}}\binom{2a}{t}\binom{\alpha(1+t)-1}{p}\binom{2\beta(1+p)-1}{l}$$

The equation (15) above represents the r^{th} moments of the NSTIIGTLE distribution, while the mean of the distribution can be calculated by setting r = 1 in (15)."

5.2. Moment Generating Function (mgf) of the NSTIIGTLE Distribution

The Moment Generating Function of x is given as

$$M_x(t) = \int_0^{\infty} e^{tx} f(x)dx \text{ ....(16)}$$

The moment generating function (MGF) of the NSTIIGTLE distribution is obtained by substituting equation (11) into equation (16), yields the following:

$$M_x(t) = \sum_{a,t,p,l=0}^{\infty} \Phi\int_0^{\infty} e^{tx}\left[e^{-\theta x}\right]^{l+1} dx \text{ ....(17)}$$

where the expansion of $e^{tx} = \sum_{z=0}^{\infty}\frac{t^z x^z}{z!}$ and using the moment-based approach described above, the mgf of the NSTIIGTLE distribution is given in equation (18) below.

$$M_x(t) = \sum_{a,t,p,l=0}^{\infty}\sum_{m=0}^{\infty}\frac{t^m\Phi\Gamma\left(m+1\right)}{\left(l+1\right)^{m+1}\theta^m m!} \text{ ....(18)}$$

5.3. Reliability function of the NSTIIGTLE Distribution

The reliability function represents the probability that a system or component continues to function without failure beyond a specified time. It measures survival over time and provides a fundamental description of longevity in reliability and lifetime modeling. It is defined as follows:

$$R(x;\alpha,\beta,\theta) = 1 - F(x;\alpha,\beta,\theta) \text{ ....} \mathbf{19}$$

$$R(x;\alpha,\beta,\theta) = 1 - \sin\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x}\right]^{2\beta}\right]^{\alpha}\right]\right] \text{.........20}$$

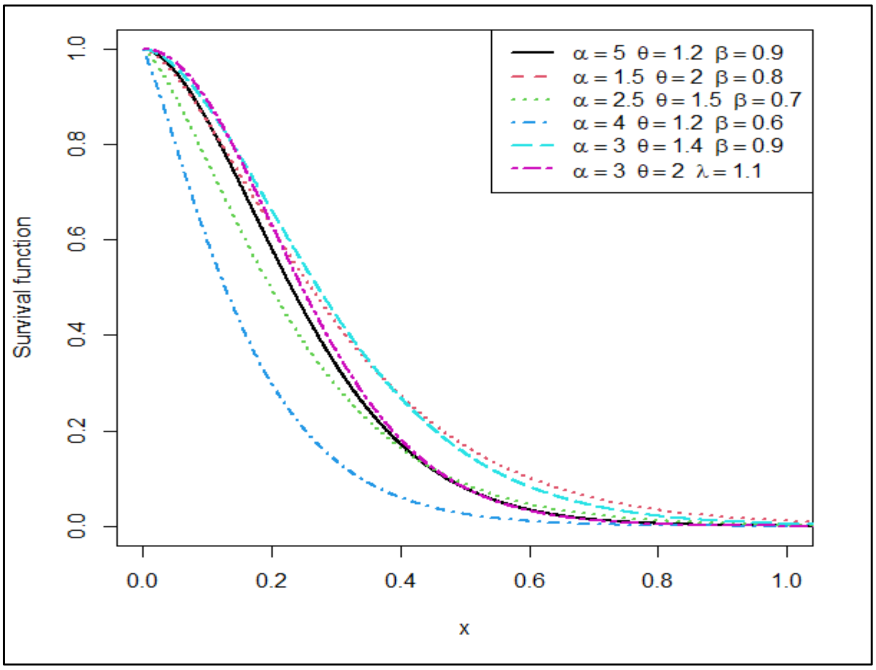

Figure 3 Plots of the survival function of NSTIIGTLE distribution with different parameter values

5.4. Hazard Function of the NSTIIGTLE Distribution

The hazard function describes the instantaneous risk that an event occurs at a particular time, given that the subject has survived up to that time. It quantifies the rate at which failures happen over time and serves as a key tool for analyzing time-to-event data in reliability and survival studies, and it is given as:

$$T(x;\alpha,\beta,\theta)=\frac{f(x;\alpha,\beta,\theta)}{R(x;\alpha,\beta,\theta)} \quad \text{......................(21)}$$

$$T(x;\alpha,\beta,\theta)=\frac{\frac{\pi}{2}2\alpha\beta\theta e^{-\theta x}\left[1-e^{-\theta x}\right]^{2\beta-1}\left[1-\left[1-e^{-\theta x}\right]^{2\beta}\right]^{\alpha-1}\cos\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x}\right]^{2\beta}\right]^{\alpha}\right]\right]}{1-\sin\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x}\right]^{2\beta}\right]^{\alpha}\right]\right]} \quad \text{......................22)}$$

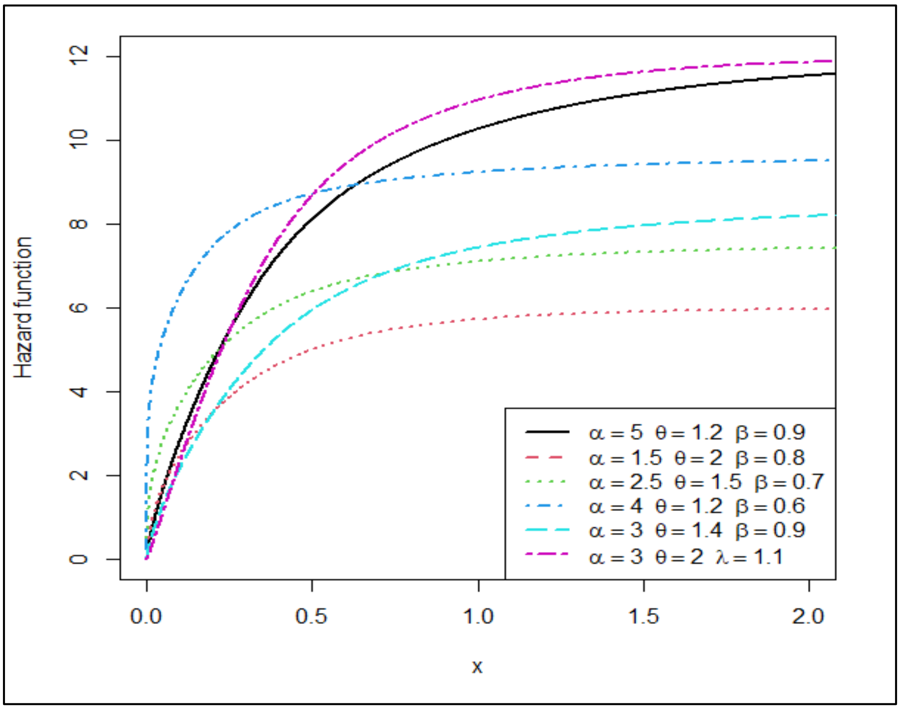

Figure 4 Plots of the hazard function of NSTIIGTLE distribution with different parameter values

5.5. Quantile Function of the NSTIIGTLE Distribution

The quantile function, also referred to as the inverse cdf, of the NSTIIGTLE distribution is derived using the cdf given in equation (5).

$$F(x;\alpha,\beta,\theta) = Sin\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x}\right]^{2\beta}\right]^{\alpha}\right]\right] = U$$

$$\left[1-\left[1-\left[1-e^{-\theta x}\right]^{2\beta}\right]^{\alpha}\right] = \frac{sin^{-1}(U)}{\frac{\pi}{2}}$$

$$\left[1-e^{-\theta x}\right]^{2\beta} = \left[1-\left[\frac{sin^{-1}(U)}{\frac{\pi}{2}}\right]\right]^{\frac{1}{\alpha}}$$

$$1-e^{-\theta x} = \left[1-\left[1-\left[\frac{sin^{-1}(U)}{\frac{\pi}{2}}\right]\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{2\beta}}$$

$$e^{-\theta x} = 1-\left[1-\left[1-\left[\frac{sin^{-1}(U)}{\frac{\pi}{2}}\right]\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{2\beta}}$$

$$-\theta x = \boldsymbol{log}\left[\mathbf{1} - \left[\mathbf{1} - \left[\frac{\boldsymbol{sin^{-1}(U)}}{\frac{\boldsymbol{\pi}}{\mathbf{2}}}\right]\right]^{\frac{\mathbf{1}}{\boldsymbol{\alpha}}}\right]^{\frac{\mathbf{1}}{\mathbf{2\beta}}}$$

$$x = \varrho(U) = \frac{1}{\theta}\left[-log\left[1 - \left[\mathbf{1} - \left[\mathbf{1} - \left[\frac{\boldsymbol{sin^{-1}(U)}}{\frac{\boldsymbol{\pi}}{\mathbf{2}}}\right]\right]^{\frac{\mathbf{1}}{\boldsymbol{\alpha}}}\right]^{\frac{\mathbf{1}}{\mathbf{2\beta}}}\right]\right] \text{..................(23)}$$

The median of the NSTIIGTLE distribution is obtained by setting u = 0:5 in equation (23), resulting in:

$$\text{median} = Q(0.5) = \frac{1}{\theta}\left[-\log\left[1 - \left[1 - \left[1 - \left[\frac{\sin^{-1}(0.5)}{\pi/2}\right]\right]^{\frac{1}{\alpha}}\right]^{\frac{1}{2\beta}}\right]\right] \text{..................(24)}$$

5.6. Distribution of Order Statistics of the NSTIIGTLE Distribution

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d) random variables with their corresponding continuous distribution function $F(x)$. Let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ the corresponding ordered random sample from the STIIGTLE distributions. Let $F_{r:n}(x)$ and $f_{r:n}(x)$, $r = 1, 2, 3, \ldots, n$ denote the CDF and PDF of the r^{th} order statistics $X_{r:n}$ respectively. The PDF of the r^{th} order statistics of $X_{r:n}$ is given as:

$$f_{rn}(x, \alpha, \beta, \theta) = \frac{f(x)}{B(r, n-r+1)} \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} [F(x)]^{v+r-1} \text{..................(24)}$$

The pdf of the rth order statistics for the NSTIIGTLE distribution is obtained by substituting equation (11) and equation (12) into equation (25), so we have:

$$f_{r:n}(x; \alpha, \beta, \theta) = \frac{\sum_{a,t,p,l=0}^{\infty} \Phi\left[e^{-\theta x}\right]^{l+1}}{B(r, n-r+1)} \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} \left[\sum_{g,w,d,s=0}^{\infty} \Upsilon\left[e^{-\theta x}\right]^s\right]^{v+r-1} \text{..................(.25)}$$

Where,

$$\boldsymbol{\Phi} = \frac{(-1)^{a+t+p+l}\pi^{2a+1} 2\alpha\beta\theta}{(2a)!\, 2^{2a+1}} \binom{2a}{t} \binom{a(1+t)-1}{p} \binom{2\beta(1+p)-1}{l}$$

and

$$\Upsilon = \frac{(-1)^{g+w+d+s}\pi^{2g+1}}{(2g+1)!\, 2^{2c+1}} \binom{2g+1}{w} \binom{\alpha w}{d} \binom{2\beta d}{S} \ldots.(26)$$

The pdf of the minimum order statistic for the NSTIIGTLE distribution is obtained by setting r = 1 in equation (26) as follows:

$$f_{1:n}(x;\alpha,\beta,\theta) = n\Upsilon\Phi \sum_{a,t,p,l=0}^{\infty} \sum_{g,w,d,s=0}^{\infty} \sum_{v=0}^{n-1} (-1)^{v} \binom{n-1}{v} \left[e^{-\theta x}\right]^{sv+l+1} \text{.................. .(27)}$$

Also, the pdf of the maximum order statistic for the NSTIIGTLE distribution is obtained by setting r = n in equation (26) as follows:

$$f_{n:n}(x;\alpha,\beta,\theta) = n\Upsilon\Phi \sum_{a,t,p,l=0}^{\infty} \sum_{g,w,d,s=0}^{\infty} (-1)^{v} \left[e^{-\theta x}\right]^{s(v+n-1)+l+1} \text{.................. (28)}$$

5.7. Renyi Entropy of the New Sine Type II Generalized Topp-Leone Exponential (NSTIIGTLE) Distribution

Entropy is a statistical metric that quantifies the level of uncertainty or randomness linked to a random variable X. For the NSTIIGTLE distribution, it is mathematically formulated as follows:

$$\mathrm{I}_R(\Upsilon) = \frac{1}{1+\Upsilon} log \int_0^{\infty} \left(f(x)\right)^{\Upsilon} dx,\ \Upsilon > 0 \ \text{ and } \Upsilon \neq 0 \text{.................. (29)}$$

By substituting equation (11) into equation (29), the entropy of the NSTIIGTLE distribution is determined as follows:

$$\mathrm{I}_R(\Upsilon) = \frac{1}{1+\Upsilon} log \sum_{a,t,p,l=0}^{\infty} \Phi^{\Upsilon} \int_0^{\infty} \left[e^{-\theta x}\right]^{l+1+\Upsilon} dx \text{..................(30)}$$

Noted

$$\int_0^{\infty} \left[e^{-\theta x}\right]^{l+1+\Upsilon} = \frac{1}{(l+1+\Upsilon)\theta}$$

So therefore

$$\mathrm{I}_R(\Upsilon) = \frac{1}{1+\Upsilon} log \left[\sum_{a,t,p,l=0}^{\infty} \frac{\Phi^{\Upsilon}}{(l+1+\Upsilon)\theta}\right] \text{..................(31)}$$

Now

$$\Phi^{\Upsilon} = \frac{(-1)^{a+t+p+l+\Upsilon}\, \pi^{2a+1+\Upsilon} (2\alpha\beta\theta)^{\Upsilon}}{(2a)^{\Upsilon}\,!\, 2^{2a+1+\Upsilon}} \left\langle \begin{matrix} \Upsilon(2a) \\ t \end{matrix} \right\rangle \left\langle \begin{matrix} \Upsilon(\alpha(1+t)-1) \\ p \end{matrix} \right\rangle \left\langle \begin{matrix} \Upsilon(2\beta(1+p)-1) \\ l \end{matrix} \right\rangle$$

6. Parameter Estimation of the NSTIIGTLE Distribution

In this section, we outline the methods for estimating the unknown parameters of the NSTIIGTLE distribution. Specifically, we explore two distinct approaches for parameter estimation.

6.1. Maximum Likelihood Estimation of the NSTIIGTLE Distribution

Let $x_1, x_2, \ldots, x_n$ represent a random sample of size n from the NSTIIGTLE distribution. The likelihood function corresponding to the vector of parameter $(a, b, q)^T$ can be expressed as:

$$logL = nlog\left(\frac{\pi}{2}\right) + nlog(2) + nlog(\alpha) + nlog(\beta) + nlog(\theta) - \theta \sum_{i=0}^{n} x_i + (2\beta - 1) \sum_{i=1}^{n} log\left[1 - e^{-\theta x_i}\right]$$

$$+(\alpha-1)\sum_{i=1}^{n} log\left[1-\left[1-e^{-\theta x_i}\right]^{2\beta}\right]+\sum_{i=1}^{n} log\left[Cos\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x_i}\right]^{2\beta}\right]^{\alpha}\right]\right]\right] \quad ....32$$

By differentiating the log-likelihood (LL) with respect to α,β,θ and equating them to zero yields:

$$\frac{\partial L}{\partial \alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n} log\left[1-\left[1-e^{-\theta x_i}\right]^{2\beta}\right]+\sum_{l=1}^{n}\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x_i}\right]^{2\beta}\right]^{\alpha}\right]$$

$$log\left[1-\left[1-e^{-\theta x_i}\right]^{2\beta}\right] Cos\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x_i}\right]^{2\beta}\right]^{\alpha}\right]\right] \quad ....33$$

$$\frac{\partial L}{\partial \beta}=\frac{n}{\beta}+2\sum_{i=1}^{n} log\left[1-e^{-\theta x_i}\right]-2(\alpha-1)\sum_{i=1}^{n}\frac{\left[1-e^{-\theta x_i}\right]^{2\beta} log\left[1-e^{-\theta x_i}\right]}{\left[1-\left[1-e^{-\theta x_i}\right]^{2\beta}\right]}$$

$$-2\frac{\pi}{2}\alpha\sum_{i=1}^{n}\left[1-\left[1-e^{\theta x_i}\right]^{2\beta}\right]^{\alpha-1}\left[1-e^{\theta x_i}\right]^{2\beta} log\left[1-e^{\theta x_i}\right] tan\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x_i}\right]^{2\beta}\right]^{\alpha}\right]\right] \quad ....34$$

$$\frac{\partial L}{\partial \theta}=\frac{n}{\theta}-\sum_{i=1}^{n} x_i+(2\beta-1)\sum_{i=1}^{n}\frac{e^{-\theta x_i}x_i}{\left[1-e^{-\theta x_i}\right]}-(\alpha-1)\sum_{i=1}^{n}\frac{2\beta\left[1-e^{-\theta x_i}\right]^{2\beta-1}e^{-\theta x_i}x_i}{\left[1-\left[1-e^{-\theta x_i}\right]^{2\beta}\right]}$$

$$-\sum_{i=1}^{n}\frac{\pi}{2}\alpha\left[1-\left[1-e^{\theta x_i}\right]^{2\beta}\right]^{\alpha-1} 2\beta\left[1-e^{\theta x_i}\right]^{2\beta-1}e^{-\theta x_i}x_i tan\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x_i}\right]^{2\beta}\right]^{\alpha}\right]\right] \quad ....35$$

Equations (33), (34) and (35) above are non-linear and cannot be solved analytically, requiring the use of computational methods obtain numerical solutions.

6.2. Maximum Product of Spacing Estimates of the NSTIIGTLE Distribution

Suppose $x_1, x_2, \dots x_n$ represents a random samples from the NSTIIGTLE distribution have CDF $F(x;\alpha,\beta,\theta)$ and $x_1, x_2, \dots x_n$ be the corresponding ordered sample. Then the spacing:

$$P_i = F\left(x_{(i)}\right)-F\left(x_{(i-1)}\right) \quad for \quad i=1,2,\dots n+1 \quad ....(36)$$

where

$$F\left(x_{(0)}\right)=0 \quad and\ F\left(x_{(n+1)}\right)=1$$

Therefore

$$F(x_{(i)};\alpha,\beta,\theta)=\sin\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x_{(i)}}\right]^{2\beta}\right]^{\alpha}\right]\right] \quad ....(37)$$

$$F(x_{(i-1)};\alpha,\beta,\theta)=\sin\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x_{(i-1)}}\right]^{2\beta}\right]^{\alpha}\right]\right] \quad ....(38)$$

Substituting equations (37) and equation (38) into equation (36), we have:

$$P_i = \left[\sin\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x_{(i)}}\right]^{2\beta}\right]^{\alpha}\right]\right]\right] - \left[\sin\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x_{(i-1)}}\right]^{2\beta}\right]^{\alpha}\right]\right]\right] \ldots\ldots\ldots\ldots\ldots\ldots(39)$$

The estimates of the parameters are obtained by maximizing equation (39)

$$T(x;\alpha,\beta,\theta) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log P_i \ldots\ldots\ldots\ldots\ldots\ldots(40)$$

$$T(x;\alpha,\beta,\theta) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log\left\{\left[\sin\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x_{(i)}}\right]^{2\beta}\right]^{\alpha}\right]\right]\right] - \left[\sin\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x_{(i-1)}}\right]^{2\beta}\right]^{\alpha}\right]\right]\right]\right\} \ldots\ldots 41$$

Differentiating T with respect to each parameter provides the parameter estimates of $\hat{\alpha}_{MPS}, \hat{\beta}_{MPS}, \hat{\theta}_{MPS}$ and solving the resulting non-linear equations gives their values.

$$\frac{\partial T(x;\alpha,\beta,\theta)}{\partial \alpha} = \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{P_i}\left[D_1(x_{(i)};\alpha,\beta,\theta) - D_2(x_{(i-1)};\alpha,\beta,\theta)\right] \ldots\ldots\ldots\ldots\ldots\ldots 42)$$

Where

$$D_1(x_{(i)};\alpha,\beta,\theta) = \cot\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x_{(i)}}\right]^{2\beta}\right]^{\alpha}\right]\right]\frac{\pi}{2}\left[1-\left[1-e^{-\theta x_{(i)}}\right]^{2\beta}\right]^{\alpha}\log\left[1-\left[1-e^{-\theta x_{(i)}}\right]^{2\beta}\right]$$

and

$$D_2(x_{(i-1)};\alpha,\beta,\theta) = \cot\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x_{(i-1)}}\right]^{2\beta}\right]^{\alpha}\right]\right]\frac{\pi}{2}\left[1-\left[1-e^{-\theta x_{(i-1)}}\right]^{2\beta}\right]^{\alpha}\log\left[1-\left[1-e^{-\theta x_{(i-1)}}\right]^{2\beta}\right]$$

$$\frac{\partial T(x;\alpha,\beta,\theta)}{\partial \beta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{P_i}\left[R_1(x_{(i)};\alpha,\beta,\theta) - R_2(x_{(i-1)};\alpha,\beta,\theta)\right] \ldots\ldots\ldots\ldots\ldots\ldots(43)$$

Where

$$R_1(x_{(i)};\alpha,\beta,\theta) = \cot\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x_{(i)}}\right]^{2\beta}\right]^{\alpha}\right]\right]\frac{\pi}{2}\alpha\left[1-\left[1-e^{-\theta x_{(i)}}\right]^{2\beta}\right]^{\alpha-1} 2\left[1-e^{-\theta x_{(i)}}\right]^{2\beta}\log\left[1-e^{-\theta x_{(i)}}\right]$$

And

$$R_2(x_{(i-1)};\alpha,\beta,\theta) = \cot\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x_{(i-1)}}\right]^{2\beta}\right]^{\alpha}\right]\right]\frac{\pi}{2}\alpha\left[1-\left[1-e^{-\theta x_{(i-1)}}\right]^{2\beta}\right]^{\alpha-1} 2\left[1-e^{-\theta x_{(i-1)}}\right]^{2\beta}\log\left[1-e^{-\theta x_{(i-1)}}\right]$$

$$\frac{\partial T(x;\alpha,\beta,\theta)}{\partial \theta} = \frac{1}{n+1}\sum_{i=1}^{n+1}\frac{1}{P_i}\left[V_1(x_{(i)};\lambda,\alpha,\theta) - V_2(x_{(i-1)};\lambda,\alpha,\theta)\right] \ldots\ldots\ldots\ldots\ldots\ldots(44)$$

where

$$\mathrm{V}_1(x_{(i)};\alpha,\beta,\theta)=\cot\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x_{(i)}}\right]^{2\beta}\right]^{\alpha}\right]\right]\frac{\pi}{2}\alpha\left[1-\left[1-e^{-\theta x_{(i)}}\right]^{2\beta}\right]^{\alpha-1}2\beta\left[1-e^{-\theta x_{(i)}}\right]^{2\beta-1}e^{-\theta x_{(i)}}x_{(i)}$$

And

$$\mathrm{V}_2(x_{(i-1)};\alpha,\beta,\theta)=\cot\left[\frac{\pi}{2}\left[1-\left[1-\left[1-e^{-\theta x_{(i-1)}}\right]^{2\beta}\right]^{\alpha}\right]\right]\frac{\pi}{2}\alpha\left[1-\left[1-e^{-\theta x_{(i-1)}}\right]^{2\beta}\right]^{\alpha-1}2\beta\left[1-e^{-\theta x_{(i-1)}}\right]^{2\beta-1}e^{-\theta x_{(i-1)}}x_{(i-1)}$$

The MPS are obtained by setting equation (42), (43) and (44) to zero and solving the resulting equations simultaneously. Since, these equations cannot be solved analytically, necessitating the use of computational techniques for numerical solution.

7. Simulation Study of the New Sine Type II Generalized Topp-Leone Exponential (NSTIIGTLE) Distribution

In this analysis, 1000 samples were simulated from the NSTIIGTLE distribution through the quantile function defined in equation (3.58). The selected sample sizes were n=20, 50, 100, 250, 500, and 1000. These simulated datasets were then employed to estimate the model parameters and to calculate the correspondence bias and Root Mean Square Error (RMSE). The summarized results are presented in Tables 4.1 and Table 4.2. Table 4.1 and Table 4.2 present the parameter estimates obtained using we the MLE and MPS methods, together with their associated bias and root mean square error (RMSE), for the NSTIIGTLE at parameter values of α = 0.4, β =1, θ=0.5, and α = 0.5, β =1, θ=1.5, respectively. The results presented in the table show that increasing the sample size leads to a progressive reduction in both bias and RMSE, with these measures tending toward zero. This pattern indicates that the estimators improve in accuracy and stability as the sample size grows. Consequently, the parameter estimates can be regarded as efficient and consistent, since larger samples yield more precise estimation.

Table 1 MLEs, MPSs, biases and RMSE for some values of parameters

		Estimation Methods					
		MLE			**MPS**		
N	**Parameters**	**Estimated Values**	**Bias**	**RMSE**	**Estimated Values**	**Bias**	**RMSE**
20	$\alpha = 0.4$ $\beta = 1.0$ $\theta = 0.5$	0.4968 1.3376 0.6380	0.0968 0.3376 0.1380	0.3612 0.7796 0.3765	0.3891 1.0106 0.6429	-0.0109 0.0106 0.1429	0.2743 0.5533 0.3365
50	$\alpha = 0.4$ $\beta = 1.0$ $\theta = 0.5$	0.4493 1.1407 0.5860	0.0493 0.1407 0.0860	0.2524 0.4153 0.2718	0.3956 0.9897 0.5816	-0.0044 -0.0103 0.0816	0.2076 0.3481 0.2429
100	$\alpha = 0.4$ $\beta = 1.0$ $\theta = 0.5$	0.4244 1.0737 0.5540	0.0244 0.0737 0.0540	0.1811 0.2312 0.1998	0.4019 0.9873 0.5482	0.0019 -0.0127 0.0482	0.1680 0.1916 0.1911
250	$\alpha = 0.4$ $\beta = 1.0$ $\theta = 0.5$	0.4164 1.0292 0.5206	0.0164 0.0292 0.0206	0.1222 0.1381 0.1388	0.4101 0.9885 0.5164	0.0101 -0.0115 0.0164	0.1217 0.1255 0.1412
500	$\alpha = 0.4$ $\beta = 1.0$	0.4107 1.0153	0.0107 0.0153	0.1003 0.1001	0.4080 0.9910	0.0080	0.0989 0.0930

	$\theta = 0.5$	0.5142	0.0142	0.1153	0.5095	-0.0090 0.0095	0.1126
1000	$\alpha = 0.4$ $\beta = 1.0$ $\theta = 0.5$	0.4112 1.0043 0.5043	0.0112 0.0043 0.0043	0.0813 0.0703 0.0892	0.4124 0.9900 0.4988	0.0124 -0.0100 - 0.0012	0.0812 0.0671 0.0870

Table 2 MLEs MPSs, biases and RMSE for some values of parameters

		Estimation Methods					
		MLE			**MPS**		
N	**Parameters**	**Estimated Values**	**Bias**	**RMSE**	**Estimated Values**	**Bias**	**RMSE**
20	$\alpha = 0.5$ $\beta = 1.0$ $\theta = 1.5$	0.6208 1.2477 1.6937	0.1208 0.2477 0.1937	0.3985 0.6216 0.6747	0.4743 0.9667 1.7144	-0.0257 -0.0333 0.2144	0.2810 0.4252 0.5843
50	$\alpha = 0.5$ $\beta = 1.0$ $\theta = 1.5$	0.5457 1.1052 1.6560	0.0457 0.1052 0.1560	0.2569 0.3268 0.5745	0.4812 0.9700 1.6387	-0.0188 -0.0300 0.1387	0.2057 0.2839 0.4944
100	$\alpha = 0.5$ $\beta = 1.0$ $\theta = 1.5$	0.5158 1.0585 1.6169	0.0158 0.0585 0.1169	0.1829 0.1976 0.4460	0.4839 0.9804 1.6017	-0.0161 -0.0196 0.1017	0.1616 0.1591 0.3995
250	$\alpha = 0.5$ $\beta = 1.0$ $\theta = 1.5$	0.5027 1.0260 1.5672	0.0027 0.0260 0.0672	0.1235 0.1130 0.3195	0.4902 0.9884 1.5551	-0.0098 -0.0116 0.0551	0.1155 0.1040 0.3034
500	$\alpha = 0.5$ $\beta = 1.0$ $\theta = 1.5$	0.4936 1.0155 1.5568	-0.0064 0.0155 0.0568	0.0898 0.0793 0.2484	0.4873 0.9947 1.5532	-0.0127 -0.0053 0.0532	0.0919 0.0755 0.2523
1000	$\alpha = 0.5$ $\beta = 1.0$ $\theta = 1.5$	0.4919 1.0095 1.5481	-0.0081 0.0095 0.0481	0.0730 0.0574 0.2021	0.4913 0.9971 1.5381	-0.0087 -0.0029 0.0381	0.0740 0.0549 0.2023

8. Application of the new models to Real-Life Datasets

To demonstrate the practical usefulness of the proposed distribution, it is fitted to real-life datasets. The goodness-of-fit of the model is compared with other related distributions using statistical measures such as, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Log-likelihood (LL) values. The datasets used in this research work are presented below.

8.1. Data Set 1

Data set 1 consists of 106 observations, representing COVID-19 mortality rates in Mexico from 4 March to 20 July 2020. The data sets has been used by Almongy *et al.,* (2021), Muhammad (2022) and Olubiyi *et al.,* (2024).

8.826, 6.105, 9.391, 14.962, 10.383, 7.267, 13.220, 16.498, 11.665, 6.015, 10.855, 6.122, 6.656, 3.440, 5.854, 10.685, 10.035, 5.242, 4.344, 5.143, 7.630, 14.604, 7.903, 6.370, 3.537, 6.327, 4.730, 3.215, 9.284, 12.878, 8.813, 10.043, 7.260, 5.985, 6.412, 3.395, 4.424, 9.935, 7.840, 9.550, 3.499, 3.751, 6.968, 3.286, 10.158, 8.108, 6.697, 7.151, 6.560, 2.077, 3.778, 2.988, 3.336, 6.814, 8.325, 7.854, 8.551, 3.228, 7.486, 6.625, 6.140, 4.909, 4.661, 5.392, 12.042, 8.696, 1.815, 3.327, 5.406, 6.182, 1.041, 1.800, 4.949, 4.089, 3.359, 2.070, 3.298, 5.317, 5.442, 4.557, 4.292, 2.500, 6.535, 4.648, 4.697, 5.459, 4.120, 3.922, 3.219, 1.402, 2.438, 3.257, 3.632, 3.233, 3.027, 2.352, 1.205, 3.218, 2.926, 2.601, 2.065, 3.029, 2.058, 2.326, 2.506, 1.923

8.2. Fitting the New Sine Type II Generalized Topp-Leone Exponential (NSTIIGTLE) Distribution

Here, the NSTIIGTLE distribution is applied to analyze data set 1. An extensive comparison is performed by fitting the NSTIIGTLE distribution against other alternative competing distributions like TIHLEtEx, ToLEx, KEx and EtEx. The comparison seeks to highlight the adaptability and appropriateness of the new distribution and assess its fit to the experimental data sets compared to the comparative models. Both Maximum Likelihood Estimate (MLE) and Method of Maximum Product of Spacing (MPS) approach are applied, and computations are carried out using R, to ensure efficiency and simplicity.

The pdf of the distributions being compared are given by:

Type I Half-Logistic Exponentiated Exponential (TIHLEtE) Distribution

$$f(x;\lambda,\alpha,\theta)=\frac{2\lambda\alpha\theta e^{-\theta x}\left[1-e^{-\theta x}\right]^{\alpha-1}\left[1-\left[1-e^{-\theta x}\right]^{\alpha}\right]^{\lambda-1}}{\left[1+\left[1-\left[1-e^{-\theta x}\right]^{\alpha}\right]^{\lambda}\right]^{2}}$$

Topp-Leone Exponential (ToLE) Distribution

$$f(x;\alpha,\theta)=2\alpha\theta e^{-2\alpha x}\left[1-e^{-2\alpha x}\right]^{\theta-1}$$

Kumaraswamy Exponential (KEx) Distribution

$$f(x;\alpha,\lambda,\theta)=\alpha\lambda\theta e^{-\alpha x}(1-e^{-\alpha x})\left[1-\left[1-e^{-\alpha x}\right]^{\alpha}\right]^{\lambda-1}$$

Exponentiated Exponential (ExEx) Distribution

$$f(x;\alpha,\theta)=\alpha\theta e^{-\alpha x}\left[1-e^{-\alpha x}\right]^{\theta-1}$$

8.3. Comparison of Results with other Competing Distributions

This section presents a comparison using the baseline exponential distribution, aiming to examine the impact of additional parameters on the distribution's applicability, effectiveness, and flexibility.

Table 3 The MLEs, Log-likelihoods and Goodness of Fits Statistics of the models based on data set 1

Model	α	θ	λ	β	LL	AIC	BIC
NSTIIGTE	0.2673	0.6331	-	2.8614	-261.6226	529.2433	537.2356
TIHLEtE	0.8333	2.1520	0.1176	-	-282.4193	570.8386	578.8289
ToLEx	1.7304	-	0.1165	-	-273.4498	550.8996	556.2264
KEx	2.1684	24.1484	0.0398	-	-263.4221	532.8443	540.8346
EtEx	0.2903	-	2.5571	-	-265.1632	534.3265	539.6534

The table 3: presents the outcomes of the Maximum Likelihood Estimation results for the parameters of the NSTIIGTLE distribution alongside four other comparative distribution. Based on the goodness of fit criteria, the NSTIIGTLE distribution attained the lowest AIC value of 529.5652, BIC value of 537.2356 and the highest LL value of -261.6226.

Consequently, among all the distributions considered, the NSTIIGTLE distribution provided the most favorable fit to the data set 1.

Table 4 The MPSs, Log-likelihoods and Goodness of Fits Statistics of the models based on data set 1

Model	α	θ	λ	β	LL	AIC	BIC
NSTIIGTE	0.3151	0.5231	-	2.2755	-261.7826	529.5652	537.5555
TIHLEtEx	0.8894	2.2417	0.1098	-	-282.5603	571.1296	579.1109
ToLEx	3.7497	-	0.1726	-	-273.8967	551.7934	557.1206
KEx	4.6120	0.7818	0.4351	-	-263.6916	533.3832	541.3735
EtEx	0.3451	-	3.7478	-	-266.8520	537.7040	540.8710

Table 4: presents the Maximum Product of Spacing Estimates of the NSTIIGTLE distribution parameters and those of four other competing distributions. The goodness-of-fit criteria used is the AIC and BIC. The NSTIIGTLE distribution achieved the lowest AIC value of 529.5652, BIC value of 537.5555 and the highest LL value of -261.7826. As a result, among all the considered distributions, showing that it provides the best fit to the daily data set 1.

9. Summary and Conclusion

9.1. Summary

This study develops the New Sine Type II Generalized Topp-Leone Exponential family of distributions and examines its theoretical and practical relevance. We presents two novel continuous probability distributions: the New Sine Type II Generalized Topp-Leone Exponential Distribution and the New Sine Type II Generalized Topp-Leone Distribution. The flexibility of the proposed family is visually demonstrated through the probability density and cumulative distribution plots presented in Figures 1 and 2, where different parameter combinations generate a wide variety of shapes. Our analysis reveals the distinct shapes of these distributions, as evident from the plots of their probability density functions (pdfs) and hazard rate functions. Notably, the shapes of these distributions are influenced by their parameter values. These distributions show promise for modelling real-world phenomena with inverted bathtub, decreasing, and increasing failure rates. We derive and examine the properties of these novel models.

Parameter estimation is investigated through a Monte Carlo simulation study, with results presented in Tables 1 and 2, showing estimated parameters alongside their corresponding biases and mean squared errors. The decreasing bias and mean squared error values with increasing sample sizes provide evidence of estimator consistency.

The practical usefulness of the proposed family is further supported by real data applications and evaluated using the MLE and MPS methods for parameters estimation as summarized in Tables 3 and 4. The goodness-of-fit statistics, including AIC, BIC values, indicate that our models consistently outperformed competing distributions.

9.2. Conclusion

This research developed a novel Sine based-based extension of the Type II generalized Topp-Leone exponential distribution to improve flexibility in modelling complex lifetime data. The structural properties of the model were carefully established, showing that it accommodates a wider range of distributional shapes. The analytical derivations of its statistical characteristics provide a solid theoretical basis for its application. The estimation procedure, implemented through maximum likelihood, and maximum product of spacing techniques, demonstrated reliable performance across different sample sizes, as supported by simulation outcomes.

Empirical applications to biomedical datasets further confirmed the strength of the model. The proposed distribution consistently delivered improve goodness-of-fit measures when compared with traditional and related models. This suggests that the model is particularly effective for capturing real-life survival patterns where conventional distributions may be too restrictive.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to be disclosed.

References

[1] Almongy H. M., Almetwally E. M., Aljohani H. M., Alghamdi A., Hafez E. H. (2021). A new extended Rayleigh distribution with applications of Covid-19 data. Results in Physics, 23, 104012.

[2] Cheng R. C. H. and Amin N. A. K. (1979). Maximum product of spacings estimation with application to the lognormal distribution Math report

[3] Cheng R. C. H. and Amin N. A. K. (1979). Estimating parameters in continuous univariate distributions with a shift origin. Journal of the Royal Statistical Society: Series B (Methodological), 45(3), 394-404

[4] Cordeiro G. M, De Castro M. (2011). A new family of generalized distribution. Journal of Statistical computation and simulation, 81(7), 883-898.

[5] Isa A. M., Doguwa S. I., Alhaji B. B., Dikko H. G. (2023). Sine Type II Topp-Leone G family of distribution: Mathematical properties and Application. Arid Zone Journal of basic and applied reseach, Faculty of Science, Borno State University Maiduguri, Nigeria. Vol. 2(4), 124-138

[6] Jones, M. C. (2004). Families of distributions arising from distributions of order statistics. TEST 13(1): 1-43

[7] Marshall A. W, and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with applications to exponential and Weibul families. Biometrika, 84(3), Pp. 641-652.

[8] Muhammad M., Alshanbiri H. M.,Alanzi A. R., Liu L., Sami W., Chesneau C., Jamal F. (2021). A newgenerator of probabilitymodels: the exponentiated sine-G family for lifetime studies. Entropy, 23(11), 1-30

[9] Nadarajah S., and Kotz S. (2006). The beta exponential distribution. Reliability engineering and system safety, 91(6), Pp. 689-697.

[10] Olubiyi A. O., Olayemi M. S., Olajide O. O., Illesanmi A. O. (2024). Development of a modified probability distribution, properties and its applications to Biomedical Datasets. International Journal of Mathematics, Statistics and Operations Research: Vol.4; Number 1; 2024, Pp49-66

[11] Rezaei S., Bahrami S. D., Alizadeh M., Nadarajah S. (2017). Topp-Leone generated family of distributions. Communications in Statistics-Theory and Mehods, 46(6), 2893-2909

[12] Topp C. W. & Leone F. C (1955). A family of J-shaped frequency functions. Journal of American Statistical Association, 50(269), 209-219.