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The dual of linear topological space

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Abstract

In this work, locally convex linear topological space $\boldsymbol{X}$ with dual space $\boldsymbol{X}^*$ are presented for the investigation of the existence and properties of various unique topologies on the dual space $\boldsymbol{X}^*$. We established the existence of three distinct topologies on $\boldsymbol{X}^*$ generated by polar of subsets of $\boldsymbol{X}$. Uses were made of the definition of base and subbase of topology and properties of locally convex Hausdorff topology determined by a separating family $\boldsymbol{P}$ of seminorms having base at zero. We have also established that the three distinct topologies on $\boldsymbol{X}^*$ are convex, absorbent and Hausdorff, using the concept of seminormed linear topology. Not alone, we established that the point open topology is less than the compact open topology while the compact open topology is less than the strong topology. And that the compact open topology coincides with the strong topology whenever $\boldsymbol{X}$ is of finite dimensional. Use was made of the definition and concepts of coarser topology.

Keywords: Topological Space; Linear Map; Convex Set; Balanced Set and Absorbent Set

1. Introduction

In modern mathematics, topology has been one of the most influential fields. Although its origin may be traced back several hundreds of years. Johan Benedict listing introduced the term topology (which are German's name) and having used the word for ten years before its first appearance in print. The English form "topology" was used in the year 1883. The term topologist for specialist in topology was used in 1905 (Taherifar, 2014).

And this work was consolidated and extended by a physician called Henri Poincare who "gave topology wings". The man called Maurice Frechet introduced the metric space in 1906, which is now a special case of a general topological space. In 1914, Felix Hausdorf f comes up with the term topological space and gave the definition for what is now called a Hausdorf f space. A dual space can be defined as, given any vector space V over a field F, the dual space V* is the set of all linear maps $\emptyset: V \rightarrow F$ which are continuous(linear functional). The dual space V^* itself becomes a vector space over F when equipped with an addition and multiplication satisfying (Ivan, 2008);

(i) $(\theta + \beta)(x) = \theta(x) + \beta(x)$

(ii) $(\alpha\theta)(x) = \alpha\theta(x)$ for all $\emptyset$ and $\beta\epsilon V^*, x\epsilon V$ and $\alpha\epsilon F$.

In other words, the collection of linear functions on a vector space is known as dual space. According to Oraekie, 2020, he defined topology as the family of open sets τ in a given set that satisfy the following three Axioms.

(i) Empty set & the set its self are in τ

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(ii) The union of any finite or infinite number of sets in τ belongs to τ.

(iii) The intersection of any finite sets in τ belong to τ.

And all the members of τ are called open sets, the pair of (X, τ) is called a topological space. While τ is a subset of X. Furthermore, branches of topology are as follows:

- a. General topology (point set topology)
- b. Differential topology
- c. Geometric Topology
- d. Algebraic topology

1.1. General Topology

In mathematics, general topology deals with the basic set-theoretic definitions and constructions used in topology and it is the foundation of other branches of topology. The fundamental concepts of general topology are continuity, compactness and connectedness. These fundamental concepts of general topology can be explained as (Ivan, 2008);

- **Continuity (Continuous Functions):** This continuity is expressed in terms of neighborhoods. For example, F is continuous at some point $x \in X$ ' if and only if for any neighborhood K of f(x)there is a neighborhood h of x such that $f(h) \subset K$. It means that no matter how small k is, there is always a h containing x that maps inside k and whose image under f contains f(x). Continuous function takes nearby points to nearby points (Dow & Mill, 2020).
- **Compactness:** These are those that can be covered by finitely many sets of arbitrarily small size. Formally, a topological space X is called compact if each of its open covers has a finite sub-cover. Otherwise, it is called non-compact. Some branches of mathematics like algebraic geometry, which was influenced by the French school of Bourbaki use the term quasi compact for the general notion and reserve the term compact for topological space that are both Hausdorff and quasi compact. Every closed interval in R of a finite length is compact. A set is compact if and only if it is closed and bounded (see Heine-Borel theorem). A compact subset of a Hausdorff space is closed and every continuous image of a compact space is compact (Ivan, 2008).
- **Connected Sets.** These are sets that cannot be divided into two parts that are far apart. A subset of a topological space is said to be connected if it is connected under its subspace topology. In other words, a topological space is said to be disconnected if it is the union of two disjoint non-empty open sets (Ivan, 2008).

1.2. Differential Topology

It is the type of topology that deals with differentiable functions. That is, a function whose derivatives exist at each point in its domain must be smooth and cannot contain any breakable line and the graph must have a tangent line at each interior point in its domain.

1.3. Geometric Topology

It is the study of topological space that locally looks like Euclidean space. Each point of an n-dimensional manifold has a neighborhood that is a continuous function between topological spaces that has a continuous inverse function to the Euclidean space of n-dimension and such continuous function between them is embedded with one manifold into another.

1.4. Algebraic Topology

This is one of the branches of mathematics that uses abstract algebra to study topological space. Its basic goal is to find the algebraic transformation that classifies topological space up to a continuous function between vector spaces that has a continuous inverse function. Now, let us consider a class of space that is endowed with both a topological space and algebraic structure. A dual topology or locally convex spaces are typical examples of topological vector spaces that generate normed space. It can be defined as topological vector space whose topology is generated by translation of balanced, absorbent and convex sets which are homeomorphism. In other words, it can also be defined as a topological vector space with a collection of semi-norms (Dow & Mill, 2020). The existence of a convex neighborhood base for zero vectors is enough for the Hahn-banach theorem to hold. Locally convex space can also be called a topological space $(X, \tau)$for which there exists at least one metric d on X such that $(X, d)$is a complete metric space and d induces the topology τ. A topological vector space is called locally convex if the origin has a neighborhood basis (i.e a local basis) consisting of convex sets. In fact, every locally convex topological vector space has a neighborhood basis of the origin consisting of absolutely convex sets, where this neighborhood basis can further be chosen to also consist entirely of open sets or entire of close sets (Naric & Beckenstein, 2011).However, numbers of topologies on dual spaces are as follows;

1.4.1. Weaker Topology:

Let τ_1 *and* τ_2 be two topologies on a set X of a dual space such that τ_1 *is* contained in τ_2. Then, the topology τ_1 is said to be weaker than τ_2 *and* τ_2 *is* said to be stronger topology than τ_1 (Munkres, 2016). If additionally, $\tau_1 \neq \tau_2$ *we* say that τ_1 is strictly weaker than τ_2 *and* τ_2 is strictly finer than $\boldsymbol{\tau_1}$. Thus, a topology with few open sets is called weak topology while the trivial topology is the weakest.

1.4.2. Trivial Topology:

let (X, τ) *be* a topological space. Then is said to be a trivial topology if and only if τ consist of only two open sets X itself and the empty set $\emptyset$. *i.e* $X, \emptyset \in \tau$ (Dow & Mill, 2020).

1.4.3. Strong Dual Topology:

Let $(X, Y, \langle , \rangle)$ be a dual pair of vector spaces over the field F *of* R *or* ₵. Let us denote α as the system of all subsets B contained in X bounded by elements of Y. In this,

$$\forall\, y \in Y \quad \sup_{x \in B} |(x, y)| < \infty.$$

Thus, the polar topology or strong topology α(Y,X) on Y is defined as the locally convex topology on Y generated by semi norms of the form $\sup_{x\in B} |(x, y)|, y \in Y, B \in \alpha$. At this point, when X is a locally convex space, the strong topology on the dual space X' is defined as the strong topology α(X',X)and it coincides with the topology of uniform convergence on bounded sets in X, this means that with the topology on X' generated by the semi norms of this

$\|f\|_B = \sup_{x\in B} |F(x)|, f \in X'$ where B runs over the family of all bounded sets in X. X' with this topology is called strong dual space of the space X and it is denoted by X'. In other words, a strong *topology on the continuous dual space a topological vector space X* is called the finest polar topology (Ivan, 2008). That is, the topology with most open sets on a dual pair.

1.4.4. ***Strongest Topology on Dual Space:*** *This is a discrete topology on a dual space.*

Thus, discrete topology is an example of topological space in which the points forms a sequence that is discontinuous, meaning that they are isolated from each other. Furthermore, the properties of the dual topology are convex, locally convex, absorbent, and Hausdorff. A subset "A" of a vector space X over a field "F" is called absorbent in X if any of the following conditions is satisfied;

- $For\ every\ x \in X,\ A\ aborsbs\ \{x\}.$
- $For\ every\ x \in X,\ \exists\, r > 0\ such\ that\ x \in cA\ \ for\ any\ scalar\ c \in F\ satisfying\ |c| \geq r.$
- $For\ every\ x \in X, \exists\, r > 0\ such\ that\ cx \in A\ for\ any\ scalar\ c \in F\ satisfying\ |c| \leq r.$
- $For\ every\ x \in X, \exists\, r > 0\ such\ that\ B_r\,(x) \subseteq\ A.$

Here,

$B_r\,(x) = \bar{B}_r(x) = \{c \in F \colon |c| \leq r\}$ is open ball of radius r in F with centre x and $B_r\,(x) = \{cx \colon c \in \bar{B}_r(x)\} = \{cx \colon c \in F\ and\ |c| < r\}$. Here, the closed ball is used in place of open ball.

(i) $For\ every\ x \in X, \exists\, r > 0\ such\ that\ B_r\,(x) \subseteq$ A∩ $Fx,\ where\ fx = span\{x\}.$

Proof

This follows from the previous conditions since $B_r\,(x) \subseteq\ Fx, so\ that$

$$B_r\,(x) \subseteq\ A\ if\ and\ only\ if\ B_r\,(x) \subseteq\ A \cap Fx$$

(ii) A contains the origin and for every 1-dimensional vector space $Y\ of\ X, A \cap Y$ is a neighborhood of the origin in Y. When Y is given its unique Hausdorff vector topology.

(iii) A contains the origin and for every 1-dimensional vector subspace Y of X,

$A \cap Y$ is absorbing in the Y.

(iv) The algebraic interior A contains the origin (i.e $0 \in i_A$) (Dow & Mill, 2020).

1.5. Statement of Problem

Let (X, τ)be a locally convex linear topological space over the scalar field K, where K is either the real number R or the complex number ₵ with the usual metric space topology. And let set X* be a family of all continuous linear functions f* on X into the scalar field K. We therefore, wish to investigate the various topologies on X* for which it is a linear topological space with a view to establish their properties and relationship amongst them.

1.6. Aim of the Study

The aim of this study is to describe how one can introduce a topology into a dual space X^* by means of families of linear functionals on X and to examine some of the basic properties of such topologies.

1.7. Specific Objectives

Our specific objectives in this research work are to cultivate the three distinct topologies on X^* as a dual space of X generated by;

- $Polars\ of\ finite\ subets\ of\ X*.$
- $Polars\ of\ compact\ subet\ of\ X*.$
- $Polars\ of\ bounded\ subet\ of\ X*$ and to establish their relationships or characteristics.

1.8. Scope of Study

This work is basically on locally convex Hausdorff topological space with emphasis on the dual of linear topological spaces.

1.9. Basic Definitions

1.9.1. Topological Spaces

Let X be a non-empty set and τ be a family of subset of X such that:

- The empty set ∅ belongs to τ.
- The whole set X belongs to τ.
- Any union of members of τ belongs to τ.
- Any finite of members of τ belongs to τ.

Then τ is called a topology on X and the pair (X, τ) becomes a topological space. Each member of τ is called τ-open set (John, 2018).

1.9.2. Neighborhood of a Point

A neighborhood of a point $x \in X$ *is any set* $E \subset X$ which contains an open set, say $u \subset X$ such that $x \in u \subset E$. We denote the collection of all neighborhoods of x *by* $N(x)$. A sub-collection ρ of $N(x)$ is called a base of neighborhoods of X or a base at x *if* and only if given a set $E \in N(x)$, there exists $u \in \rho$ such that $u \subset E$ (John, 2018).

1.9.3. Coarser or Weaker Topology

Let τ_1, τ_2 be a topologies on a given set X such that every $\tau_2 -$ open set is $\tau_1 -$ open set, then τ_1 *is said to be finer than* τ_2. *Or* τ_2 is coaser or weaker than τ_1 (Dow & Mill, 2020).

1.9.4. Metric Spaces

Let X *be a non − empty set and l* et R denote the set of real numbers. A metric d on a set X is a real valued function $d: XxX \to R$ *from the product space XxX into R* which satisfies the following conditions for x, y *and* $Z \in X$,

- $d(x, y) \geq 0\ \forall\ x, y$ *non negative*

- $d(x,y) = 0\ if\ and\ only\ if\ x = y$
- $d(x,y) = d(y,x)\ \ symmetric.$
- $d(x,y) \leq d(x,z) + d(z,y)\ \ triangle\ inequality.$

Condition (iii) is called the symmetric property of the metric and the condition (iv) is called the triangular inequality. A metric space consist of two objects namely, a non-empty set X and a metric d on X and we denoted this by (X,d) (John, 2018).

1.9.5. Open Sphere

Let $(X,d) be\ a\ metric\ space. Given\ a\ point\ \ x_0 \in X\ and\ a\ positive\ number\ r > 0, then\ the\ set$

$B_r(\ x_0, r) = \{y \in X: d(\ x_0, y) < r\}$ is called an open sphere centered at $x_0\ of\ radius\ r. A\ linear\ space\ over\ the\ real\ or$ complex number field k is a nonempty set X in which the two operations, namely;

(a) Addition and

(b) Scalar multiplication are defined i.e. for every $a, b\ \in X, a + b \in X. and\ for\ every\ \Lambda \in R\ and\ x \in X,\ \Lambda x \in X. Such\ that$

$(i)\ The\ set\ X\ is\ an\ additive\ abelian\ group.$

$(ii)\ for\ every\ \Lambda, \mu \in K\ and\ x, y \in X,$

$$\Lambda(x + y) = \Lambda x + \Lambda y$$

$$(\Lambda + \mu)x = \Lambda x + \mu x$$

$$(\Lambda\mu)x = \Lambda(\mu x) = \Lambda\mu x$$

$$1x = x, where\ 1\ is\ the\ identity\ in\ k\ \text{(John, 2018).}$$

1.9.6. Linear Map

Let $X_1, X_2\ be\ linear\ space\ over\ K. a\ mapping\ F:\ X_1 \to\ X_2\ is$ said to be a linear map if

$F\ (\alpha x + \beta y) = \alpha F(x) + \beta F(y) for\ every\ x\ , y \in X\ and\ \alpha, \beta \epsilon K.$ We denote the collection of all linear maps from $X_1\ into\ \ X_2\ by\ L\ (\ X_1,\ X_2)$(Swartz, 2012).

1.9.7. Convex Set

Let X be a linear space over K. a subset E of X is called convex if $\Lambda x + \mu y\ \in E\ for$ every pair x,y of points of E and every pair $\Lambda, \mu \in K, \mu, \Lambda > 0\ such\ that\ \mu + \Lambda = 1$ (John, 2018).

1.9.8. Balanced Set

If X is linear space over K, a subset B of X is balanced if and only if for $\Lambda \in K,\ 0 < |\Lambda| \leq 1,$ we have tha t $\{\Lambda x\ :\ x \in B\} \subset B\ i.e. \Lambda B \subset B$(John, 2018).

1.9.9. Absolutely Convex

A subset A of a linear space X is absolutely convex if and only if it is convex and balanced (John, 2018).

1.9.10. . Absorbent Set

A set E in a linear space X is absorbent if to each x∈X there corresponds some $r > 0$ such that $x \in \alpha E$, whenever $r \leq |\alpha|$. It is clear from our definition of absorbent that an absorbent set is balanced.

A set E absorbs a set B if and only if there exist $r > 0\ such\ that$

$$B \subseteq \alpha E\ for\ all\ \alpha\ with\ |\alpha| \leq |r|\ \text{(John, 2018).}$$

1.9.11. Linear Topological Spaces

A linear topological space (X, τ) is a linear space X over K with a topology τ such that ADDITION and MULITIPLICATION are each continuous simultaneously in both variables. That is, each of the following maps is continuous:

(i) The map $F: XxX \to X \; of \; the \; product \; XxX, with \; the \; product \; XxX$, with the product topology into X given by $F(x, y) = x + y$ is continuous.

(ii) The map $g: KxX \to X$ of the product KxX with the product topology into X, given by

$g(\lambda x) = \lambda x$. Such that $x \in X \; and \; \alpha \in k$ is continuous (John, 2018).

1.9.12. Bounded Set

A subset A of a linear topological space is called bounded if and only if for any neighborhood v of 0, there is a real number λ such that A⊂λv. Using our earlier terminology, a set is bounded if it is absorbed by every balanced neighborhood of 0 (Jackson & Naidu, 2016).

2. Main Result

$The \; weak^* \; Topology$

2.1. Preamble

One can hardly understand the dual of a linear topological space X, without being familiar with the properties and definitions of semi norm and the norm of a linear mapping, norm and linear functionals on X as our main work hinges on them.

2.2. Seminorms

Let X be a linear space over k. a semi norm on X is a real-valued function q defined on X such that;

(1) q(x)$\geq 0, \forall x \in X$.

(2) q(x+y)$\leq q(x) + q(y), \forall x, y \in X$.

(3) q$(\alpha x) = |\alpha| q(x), \forall x \in X \; and \; \alpha \in K$.

If q has the additional property $q(x) = 0 \; if \; and \; only \; if \; x = 0$. Then q is said to be a norm on X.

2.2.1. Properties of Seminorm on X.

(i) If q is a seminorm on X, then

$$|q(x) - q(y)| \leq q(x - y), \forall x, y \in X.$$

Proof

For any $x, y \in X$, we have $q(x) = q(x - y + y) \leq q(x - y) + q(y)$.

$\Rightarrow q(x) - q(y) \leq q(x - y) \ldots \ldots \ldots (1)$

Interchange $x \; and \; y$;

(i) ⇒q(y)-q(x) ≤ q(y-x)

Hence,

$$|q(x) - q(y)| \leq q(x - y).$$

(ii) If q is a seminorm on $X \; (X = linear \; space) then \; for \; any \; r > 0, r \in k$, the sets

$$B_q(x,r) = \{x \in X: q(x-0) < r\} = \{x \in X: q(x) < r\}.$$

$\bar{B}_q(x,r) = \{x \in X: q(x-0) \leq r\} = \{x \in X: q(x) \leq r\}$ are absolutely convex and absorbent.

Proof

Clearly,

$B_q(x,r) = \{x \in X: q(x) < r\} = \{x \in X: -r < q(x) < r\}$. Is an open interval containing zero.

thus, $B_q(x,r)$ is balanced. Now, if $x, y \in B_q(x,r)$ *and* $ʎ, \alpha$ are scalars such that $|ʎ| \leq 1$,

$|\alpha| \leq 1$ *and* $ʎ+\alpha = 1$. then, $q(ʎx+\alpha y) \leq |ʎ|q(x) + |\alpha|q(y) < |ʎ| + |\alpha| = 1.$ *for* $ʎ = 1 - \alpha$

this show that $B_q(x,r)$ is convex.

(iii) r$B_q(x,1) = \{x \in X: q(x-0) < r(1)\} = \{x \in X: q(x) < r\}, r \in k$.

$= B_q(0,r)$ … … … … … … . . (2).

This equation (2) showed that the right hand side of it is convex because $B_q(0,1)$ *is g* iven as $B_q(x,r) = \{x \in X: q(x) < r\}$ *is* convex.

(iv) *If* $q(y) = ʎ$, *then* $y \in (ʎ + \varepsilon)B_q(0,1), \forall \varepsilon > 0$.

Proof

$$(ʎ + \varepsilon)\, B_q(0,1) = \{x \in X: q(x) < (ʎ + \varepsilon)\} = -(ʎ + \varepsilon) < q(x) < ʎ + \varepsilon.$$

If $ʎ = 2$ *and* $\varepsilon = 1$,

so that $|ʎ| > |\varepsilon|$ by definition then, $2 \in (ʎ + \varepsilon)B_q(0,1) = \{x \in X: q(x) < (ʎ + \varepsilon)\} = \{x \in X: q(x) < 3\}$

$$= (-3 < q(x) < 3) = (-3,3) \text{ an inverse image of } q(x)$$

$$\Rightarrow 2 \in (3)B_q(0,1) = \{x \in X: q(x) < 3\} \text{ contains } 2 \text{ and hence } ʎ.$$

Showing that $B_q(0,1)$ *and hence* $B_q(0,r)$ is absorbent.

Proposition 2.2.1

Let q be a seminorm on the linear topological space X. *Then* q considered as a map from X *into* R *is* continuous if and only if $B_q(0,1)$ is a neighborhood of zero in X.

Proof

If q is continuous, then the set $B_q(x,1) = \{x \in X: q(x) < 1\}$ is clearly open being the inverse image by q of the open set $(-1,1)$ *in* R. Conversely,

If $B_q(0,1)$ *is a neighborhood of zero, so is* $rB_q(0,1) = \{x \in X: q(x) < r) \; \forall \; r > 0$.

This implies that q is continuous at origin and hence everywhere by the relation,

x+$B_q(0,1) = \{y \in X: q(x-y) < 1\}$.

2.2.2. Norm of a Function

Let X and Y be normed linear spaces. Let $F: X \rightarrow Y$ be a linear map. The map is said to be bounded if there exists a positive real number α such that

$$\|f(x)\| \leq \alpha\|x\|, \forall x \in X \ldots \ldots \ldots (a)$$

The inequality (a)shows that a bounded linear function maps bounded sets in X onto bounded sets $in\ Y$. we can cultivate the smallest possible $\alpha > 0$ such that the inequality (a) holds for all $x > 0$ and $x \in X$. $From\ (a)\ we\ have$

$$\frac{\|f(x)\|}{\|x\|} \leq \alpha, x \neq 0 \ldots \ldots \ldots .. (b)$$

If we put $\|f\| = sup \frac{\|f(x)\|}{\|x\|} \ldots \ldots \ldots .. (c)$

For every $x \in X, x \neq 0$ and $\|f\|$ is called the norm of the map or function f.

Theorem 2.2.1

Let X, Y be normed linear spaces and $F: X \rightarrow Y$ is a linear map from X $into$ Y. Then

(1) $F\ is\ continuous\ if\ and\ only\ if\ \ F\ is\ bounded.$

(2) $If\ F\ is\ continuous\ at\ a\ single\ point, it\ is\ continuous\ every\ where.$

Proof

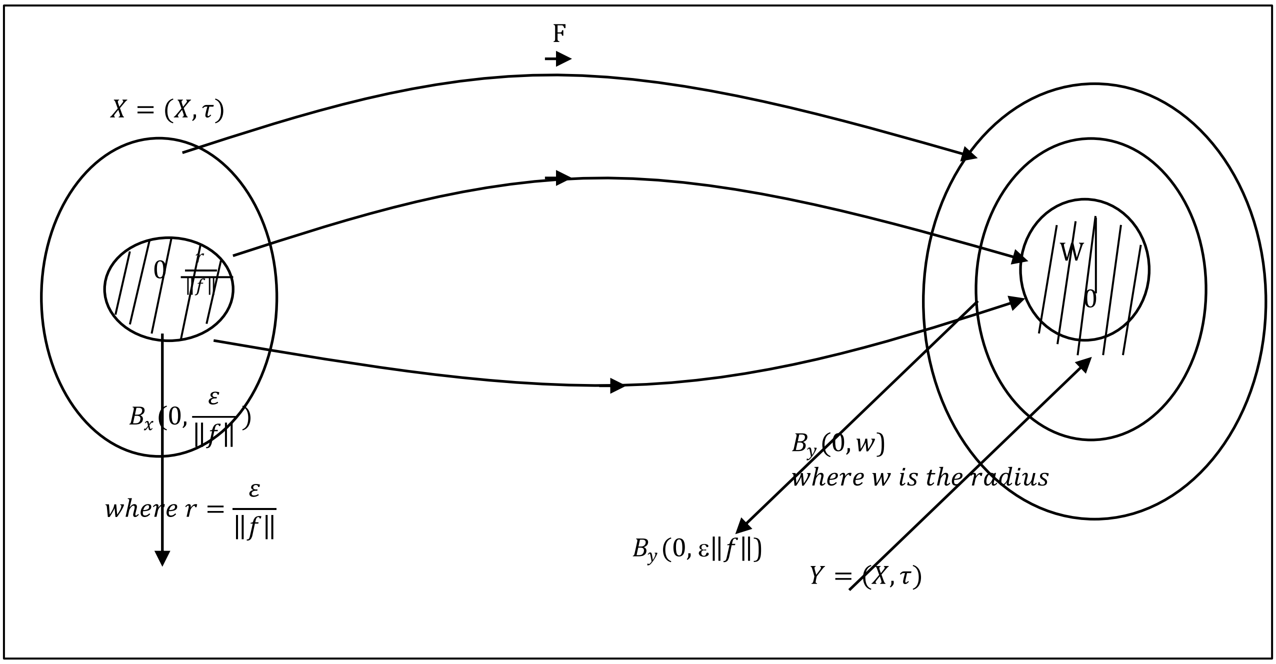
Let f be bounded linear mapping. We need to show that f is continuous at $x = 0$.

Let $B_x(0, \varepsilon) = \{x \in X: \|x\| \leq \varepsilon\}, and\ B_y(0, \varepsilon) = \{y \in Y: \|y\| \leq \varepsilon\}$

Obviously,

$$f(B_x(0, \varepsilon)) \subseteq \ (B_y(0, \varepsilon|f|)$$

$$And\ so\ f(B_x\left(0, \frac{\varepsilon}{\|f\|}\right) \subseteq B_y(0, \varepsilon)\ showing\ that\ f\ is\ continuous\ at\ 0.$$



To deduce continuity of f *at any point* $x_0 \in X, x \neq 0$, we recall that

$\{x_0 \in X: \|x - x_0\| < \varepsilon\} = x_0 + \{x \in X: \|x\| < \varepsilon\}$ (*see Oraekie* 1996).Which shows that the translate of an open ball is an open ball.

Proposition 2.1.2

If X *and* Y are normed linear topological spaces and $F: X \to Y$ *is a* bounded linear map if and only if;

(1) *It maps any bounded set onto bounded*.

(2)*If* $\|f\| = \sup_{x \in X} \frac{\|f(x)\|}{\|x\|} < \infty$

(3) *The map* $f: X \to Y$ is continuous if and only if the inverse image of any open set is open.

Proof

$(\mathbf{1}) \Rightarrow (\mathbf{2})$

Suppose that f maps a bounded set onto a bounded set. Since the unit ball given as

$B_x(0,1) = \{x \in X: \|x\| \leq 1\}$ in X is bounded,we have

$$\sup_{x \in B_x} \|f(x)\| = \sup_{\|x\| \leq 1} \|f(x)\| < \infty$$

*s*howing that (1) ⇒ (2).

(2) ⇒ (3)

Let f be a linear map which is bounded. Let $B_x(0, r) = \{x \in X: \|x\| < r\}$.

Then, $F(B_x(0,1)) \subseteq B_y(0, m + \varepsilon), \varepsilon, m \in R$

$$\Rightarrow f(B_x\left(0, \frac{1}{m + \varepsilon}\right)) \subseteq B_x(0,1)$$

$\Rightarrow f(B_x\left(0, \frac{\varepsilon}{m+\varepsilon}\right)) \subseteq B_y(0, \varepsilon)$

Proving that f is continuous at 0. Thus (2) ⇒ (3).Conversely,

if $f(B_x(0, \varepsilon)) \subseteq B_y(0, r)$, then $f(B_x(0,2)) \subseteq (B_y\left(0, \frac{2r}{\varepsilon}\right)$ *and*

$\sup_{\|x\| = 1}\{\frac{\|g(x)\|}{\|x\|}\} \leq \frac{\rho}{\varepsilon}$ *showing that* (3) ⇒ (2)

(3) ⇒ (1).

Let f be a continuous linear map.Let B be a bounded set in X, then

$B \subseteq \alpha B_x(0,1)$ *and so* $f(B) \subseteq f\left(\Lambda B_x(0,1)\right) \subseteq B_x(0, |\Lambda| \|f\| + \varepsilon)$ where ε is a small as possible. Showing that (3) ⇒ (1).

2.3. Dual Topological Space

This section is devoted to the duality which is the central part of the theory of linear topological spaces. The pattern of investigation is simple.

We wish to find, for each proposition about a linear topological space X, an equivalent proposition which is stated in terms of the adjoint space X^*.We consider two arbitrary linear spaces X *and* Y *and* a bilinear functional (the pairing functional) on their product XxY.

The weak topologies for X *and for* Y are the weakest topologies which make the pairing functional continuous in each variable separately (Kelly & Isaac, 1976) the geometry of a pairing is investigated by means of pollars, where the pollar of a subset E *of* X *is* the set of all functions f in Y *such that* $|f(x)| < 1$ *for all* x *in* $E \subset X$. In this chapter, we shall investigate topologies on linear spaces that are generated by families of linear functionals, and our investigation is neither definitive nor exhaustive. We shall specialize to two particular ones;- the weak and $weak^*$ topologies. These two topologies, the first on X and the second on dual space X^*,

are generated respectively by the continuous linear functional on X and by the continuous determined by linear functionals on X^*, that are determined by the elements of X. i.e. $T(x)$.

2.3.1. F-Topologies

Our aim here is to describe how one can introduce a topology into a linear space X by means of families of linear functionals on X and to examine some of the basic properties of such topologies. To bring the information home, suppose X is a linear space over a scalar field K and $F \subset X'$ be a family of linear functionals on X. We want to define a topology $\daleth_f$ *on* X such that $(X, \daleth_f)$ is;

(i) A locally convex topological linear space for which the functionals in

F are continuously linear functionals on $(X, \daleth_f)$.

(ii) For which a net $\{x_n\} \subset X$ converges to $x \in X$ in the topology $\daleth_f$ if and only if

$$\lim_{n \to \infty} f'(x_n) = f'(x);\ f' \in F.$$

Defining a topology $\daleth_f$ on X that satisfies the above conditions require the knowledge of topologies in seminormed linear spaces (Larsen, 1965). Using Larsen like arguments, we define a family $P = \{p_{f'} : f' \in F \subset X'\}$of seminorms on X by setting

$p_{f'}(x) = \{|f'(x)|, if\ x \in X\ and\ f' \in X'\}$. It is easily seen that each $p_{f'}$ is a seminorm.

Proof

According to Oraekie, 2014, "Given a vector space X, a seminorm (pseudo-norm, pre-norm)on X *is a map* $q: X \to R$ which satisfies the following axioms;

(i) $q(\Lambda x) = |\Lambda| q(x) for\ all\ x \in X\ and\ all\ \Lambda \in R$.

(ii) $q(x + y) \leq q(x) + q(y). for\ all\ x, y \in X$. Then, we wish to show that $p_{f'}(x) = \{|f'(x)|\},\ for\ all\ x \in X\ and\ f' \in X'$ satisfies the seminorm axioms.

Verifying axiom (i), we have

$P_{f'}(\Lambda x) = |f'(\Lambda x)| = |\Lambda||f'(x)| = |\Lambda| P_{f'}(x), \Lambda \in R, x \in X$. Verifying axiom (ii), we have

$P_{f'}(x + y) = |f'(x + y)| = |f'(x) + f'(y)|$

$\leq |f'(x)| + |f'(y)| = P_{f'}(x) + P_{f'}(y)$

$$\Rightarrow P_{f'}(x + y) \leq P_{f'}(x) + P_{f'}(y), x, y \in X.$$

Thus, we have shown that $P_{f'}(x) = |f'(x)|$ *is a seminorm*. Then the topology $\daleth_{\boldsymbol{p}}$ generated by P that is, the topology whose neighborhood has a base at a point $x \in X$ consist of sets of the form; $U(x; \varepsilon;\ P_{f_1'}, P_{f_1'}, \ldots, P_{f_n'}) = \{y: y \in X, P_{f_k'}(x - y) < \varepsilon; k = 1,2,3 \ldots, n\}$.

Where $\varepsilon > 0, n \in Z\ and\ n > 0\ and\ the\ choice\ of\ \ f_1', f_2', \ldots, f_n'\ in\ F = X'$ is arbitrary; and is such that $\{x_n\}$ converges to $x\ in\ \ \daleth_p$ if and only if

$$\lim_{n \to \infty} f'(x_n) = \ f'(x), f' \in X',$$

Assuming $|f'(x)| = P_{f'}(x) = 0, f' \in X' \Rightarrow\ x = 0$. Note that the elements of X' are not all continuous with respect to $\daleth_{\boldsymbol{p}}$. However, the pair$(X, \daleth_{\boldsymbol{p}})$ may not be a seminormed linear space or not a locally convex topological linear space, since we have not made a requirement that

$|f'(x)| = P_{f'}(x) = 0, f' \in X'. Should\ imply\ that\ x = 0$. If this requirement is done, then we have to prove the following assertions;

Proposition 2.2

Let (X, P) be a seminormed linear space over K. Then the family of open sets determined by

$$u_p(x) = \{u(x; \varepsilon; p_1, p_2, \ldots, p_n)\}$$

$= \{y: y \in X: p_k(y - x) < \varepsilon; k = 1,2 \ldots n\}\ where\ p_1, p_2, \ldots, p_n$ are any n seminorms in P.

Furthermore, for every $x \in X$, we put

$u_p(x) = \{u((x; \varepsilon; p_1, p_2, \ldots, p_n)/\varepsilon > 0; n \in Z, n > 0; p_1, p_2, \ldots, p_n \in P\}\ and$

$$U_p = \bigcup_{x \in X} u_p(x).$$

Then u_p is a base for a Hausdorff topology on X.

Proof

We need to show that U_p is a base for the Hausdor$ff\ t$oplogy on $X\ if$ and only if U_p has the following properties:

$(a)\ X = \bigcup_{x\in X} u_p(x).$

(b) $For\ any\ \cup_{p_1}, \cup_{p_2} \in \cup_p\ the\ set\ \cup_{p_1} \cap \cup_{p_2} is\ a\ union\ of\ members\ of\ \cup_p$

Recall: if $u_p\ is\ a\ basis\ for\ a\ topology\ \daleth_p\ then\ \daleth_p$ must have the properties (a),(b) and (c)of the definition of topology. In particular X must be an open sets and the intersection of any two open sets must be an open set. As the open sets are, just the union of members of U_p, this implies that (a) and (b) above are true.

Conversely, assuming that $U_p\ has\ properties\ (a) and\ (b) and\ let\ \daleth_{\boldsymbol{p}}$ be the collection of all subsets of X which are unions of members of u_p.

We shall show that $\daleth_{\boldsymbol{p}}$ is a topology on X. (if so then U_p is obviously a basis for this topology $\daleth_{\boldsymbol{p}}$ and the assertion is true). By (a)

$$X = \bigcup_{x\in X} u_p(x)\ and\ so\ X\ \in \daleth_p.$$

Note that $\emptyset\ is$ an empty union of members of $U_p\ and\ so\ \emptyset \in \daleth_p$. So we see that $\daleth_{\boldsymbol{p}}$satisfies property (i) of definition of topology of any arbitrary non-empty set X.

Now, let $\{\daleth_{\boldsymbol{p}}\}$be a family of members of $\daleth_{\boldsymbol{p}}. Then\ each\ \ \daleth_{\boldsymbol{p_j}}\ \ is$ a union of members of U_p. Hence the union of all the $\daleth_{\boldsymbol{p}}$ is also a union of members of $U_p\ and\ so\ is\ \ \daleth_{\boldsymbol{p}}. Thus, \daleth_{\boldsymbol{p}}$ also satisfies condition (ii) of the definition of topology on arbitrary set X. Finally, let $C\ and\ D\ be\ in\ \ \daleth_{\boldsymbol{p}}. We\ need\ to\ verify\ that\ C \cap D \in \daleth_{\boldsymbol{p}}.$

$$But\ C = \left(\bigcup_{j\in\Delta} u_{pj}\right)$$

$$for\ some\ index\ set\ K.\ and\ also,$$

$$D = \left(\bigcup_{j\in\Delta} u_{pj}\right),$$

$for\ i \in \Delta\ and\ u_{pj} \in U_p.$

$Therefore,$

$$C \cap D = \left(\bigcup_{j\in k} u_{pj}\right) \cap \left(\bigcup_{j\in\Delta} u_{pj}\right) = \bigcup_{j\in k, j\in\Delta} (u_{pj} \cap u_{pj}). De\ morgans\ law.$$

Observation

You should verify that the two expressions fo$r\ C \cap D\ are\ indeed\ equal$. In the finite case, this involves statements like

$$(u_{p_1} \cup u_{p_2}) \cap (u_{p_3} \cup u_{p_4})$$

$$= (u_{p_1} \cap u_{p_3}) \cup (u_{p_1} \cap u_{p_4}) \cup (u_{p_2} \cap u_{p_3}) \cup (u_{p_2} \cap u_{p_4})$$

By our assumption (b), each $u_{pj} \cap u_{pk}\ \ is$ a union of members of $u_p\ and\ so\ C \cap D$ is a union of members of U_p. Thus, $C \cap D \in \daleth_{\boldsymbol{p}}$. $so\ \ \daleth_{\boldsymbol{p}}$ has satisfied the property $(iii)$$of$ the definition of topology on set X.$Hence\ \ \daleth_{\boldsymbol{p}}$ is indeed a topology, and U_p is a basis for this topology, as required. Now, to show that the topology $\daleth_{\boldsymbol{p}}\ \ is$ Hausdorff.Suppose $x, y \in X, x \neq y$, then there exists some $p \in P\ such\ that\ p(x-y) > 0, as\ (X,p)$ is a seminormed linear space. A straightforward computation reveal that $U_p(x,\varepsilon,p) and\ U_p(y,\varepsilon,p)a$ re disjoint open neighborhood of $x\ and\ y$ respectively, $if\ we\ put$

$\varepsilon = \frac{p(x-y)}{3} = \frac{\delta}{3}$ i.e. $u_p\left(x,\frac{\delta}{3},p\right) \cap u_p\left(y,\frac{\delta}{3},p\right)$=∅.

Then from this proposition 1, we could deduce that $(X,\mathfrak{I}_p)$ is a locally convex linear topological space with the desired properties and X' has the indicated property if and only if X' separates points

Remark/ Theorem 2.3.1

Let X be a linear space over K and suppose $f \subset X^*$ separates points. If P= $\{P_{f^*}\ : f^* \in F \subset X^*\} where\ P_{f'}(x) = |f^*(x)|$, $x \in X, f^* \in F$. Then P is a family of seminorms on $X\ such\ that\ (X,p)i$ s a seminormed linear space over k. moreover, $if\ \ \mathfrak{I}_f$=$\mathfrak{I}_p$ is the topology on X determined by the family p, then

(a) $(X,\mathfrak{I}_f)\ is\ a\ locally\ convex\ linear\ topological\ space\ over\ k$.

(b) $A\ net\ \{x_n \subset X\ converges\ to$ x∈ $X\ in\ \ \mathfrak{I}_f\ \ if\ and\ only\ if$

$\lim_{n\to\infty} f^*(x_n) = f^*$(x), $f^* \in F$.

(c) $If\ f^* \in F, then\ f^*\ is\ continuous\ linear\ functional\ on\ (X,\mathfrak{I}_f)$

(d) $\mathfrak{I}_f\ is\ the\ weakest\ topology\ on\ X\ for\ which\ the\ elements\ of\ F{\subset}X'\ are\ continuous$.

Then, we refer to the topology $\mathfrak{I}_f$ constructed in this way as the $F-topology$. Or topology generated on $X'\ by$ family of linear continuous functionals on X.

2.3.2. The Weak And {Weak Topology (W*)}*

There are two particular $F-topologies\ namely$

(1) $The\ weak\ topology\ and$

(2) $The\ weak^*\ topology$.

Definition I (Weak Topology On X)W

Let (X,τ) be a locally convex topological linear space over K. Then the F —topology on X corresponding to $F = X^* \subset X'\ i$s called the weak topology on X. In this case,

$$\mathfrak{I}_F = \mathfrak{I}_{w^*}$$

Polar Sets

Definition Ii {Weak}* Topology on X').W*

Let $(X,\mathfrak{I})$ be a Banach space and $X'\ its\ dual. For\ every\ f' \in X'$, we define the seminorm

$$p_x(x) on\ X'\ by$$

$$p_x(f') = |f'(x)|.$$

The locally convex topology on $X'\ d$efined the family p_x , $x \in X$ of seminorms is called the $weak^*\ topology$.

Definition 2.3.1

Let X be a locally convex linear topological space and X' $i$ts dual (the family of all continuous linear functionals on X). Let E be the family of all finite subsets of X and E^o denote the family of polars of members of E. then

$$X^* = \bigcup_{A^o \in E^o} A^o \dots\dots\dots (i)$$

Then $\{A^o : A \in E\}$ *constitutes a subbase for a unique topology* w^* *on* X^*. This topology is an $F-$topology called the *weak** *topology on* X^*.

Proposition 2.3.2

Let (X, τ) be a locally linear topological space over a scalar field K *and* X^* its dual. Let E be the family of all finite subsets of X. then;

(i) The family of all sets of the form $\{A^o : A \in E\}$ is a local subbase for the

weak* *topology* w^*.

(ii) The space (X^*, w^*) is a Hausdorff locally convex linear topological space.

(iii) A net $\{f_n{}^*\} \subset X^*$ converges in the *weak** *topology* w^* if and only if

$$\lim_{n\to\infty} f_n{}^*(x) = f^*(x), \forall x \in X \text{ and } f^* \in X^*.$$

Proof (I).

Let $\mathcal{A} = \{A^o : A \in E\}$.

We need to show that the class β of finite intersections of members of A satisfies the two conditions for it to be a base for a topology w^* *on* X^*.

(a) $X^* = \cup \{B^o : B \in \beta\}$.

(b) *for any* $G, H \in \beta, G \cap H$ *is the union of members of* β.

observe that $X^* \in \beta$, since X^* by definition is the empty intersection of the members $\mathcal{A}$; *so*

$$X^* = \cup \{B^o : B \in \beta\}.$$

Furthermore, If $G, H \in \beta$, *then* G *and* H are finite intersections of members of $\mathcal{A}$. Hence , $G \cap H$ is also a finite intersection of members *of* $\mathcal{A}$ and therefore, belongs to β. Accordingly, β is the base for the topology w^* *on* X^* *for which* $\mathcal{A}$ is a subbase.

Proof for (2)

We need to show that (w^*, X^*) is a Hausdorff and convex. *Recall*: the open set in X^* *is of the form*

$U(x, f_0{}^*, \varepsilon) = \{f^* \in X^* : |f^*(x) - f_n{}^*(x)| < \varepsilon\}$

For $x \in X$ *and* $\varepsilon > 0$. *It is a neighborhood of* $f_0{}^*$ *in* w^*. Now, let $f_1{}^*, f_2{}^* \in X^*$ *be distinct elements of* X^*.

i.e. $f_1{}^*, f_2{}^* \in X^*$ *such that* $f_1{}^* \neq f_2{}^*$.

$$\Rightarrow \text{ there exist } x \in X \text{ such that; } f_1{}^*(x) \neq f_2{}^*(x) \text{ and}$$

$$U\left(x, f_1^*, \frac{\varepsilon}{3}\right) \text{ and } U\left(x, f_2^*, \frac{\varepsilon}{3}\right) \text{ are two distinct open neighborhoods of } f_1^* \text{ and } f_2^* \text{ respectively.}$$

Then,

$$U\left(x, f_1^*, \frac{\varepsilon}{3}\right) \cap U\left(x, f_2^*, \frac{\varepsilon}{3}\right) \neq 0.$$

If and only if since $\varepsilon = \frac{|f_1^*(x) - f_2^*(x)|}{3}$. *Showing that* w^* *is Hausdorff*. It remains to show that w^* *is convex*. This is done by show that each

A^0 is Convex.

Now, let $f_1^*, f_2^* \in A^0$ *such that* $x \in A$. *And* $\alpha > 0$ *such that* $\alpha \in (0,1)$.

$|\{\alpha f_1^* + (1-\alpha) f_2^*\}(x)| = |\alpha f_1^*(x) + (1-\alpha) f_2^*(x)| < \alpha + (1-\alpha) = 1.$

$\Rightarrow \alpha f_1^* + (1-\alpha) f_2^* \in A^0$. So A^0 *is convex*.

A^0 is Balanced

Put $\alpha \leq 1$ *and* $f^* \in A^0$ *such that* $x \in A$. $|\alpha f^*(x)| = \alpha |f^*(x)| < 1$. *Showing that* A^0 *is*

absolutely convex.

For (3)

Firstly let $f^*(x)$ *be a limit point of* X^*. Define a directed set D by

$D = \{U : U$ *is a neighborhood of* $f^*(x)\}$ *where* $U_1, U_2 \in D, U_1 \leq U_2$. As a limit point of $X^*, U \cap X^* \neq \emptyset$ *for all* $U \in D$. *For each* $U \in D$, *let* $f_u^*(x)$ *be an* arbitrary point *of* $U \cap X^*$. *Then* $\{f_u^*(x)\}$ *is a net*. If V is any neighborhood of $f^*(x)$, *then* $v \in D$.

for $u \geq v, f_u^*(x) \in U \leq V$ *so we have* $\{f_u^*(x)\} \rightarrow f^*(x)$

$$\Rightarrow \lim_{n \to \infty} f_u^*(x) = f^*(x).$$

Conversely,

Assume that $f^* \in X^*$ *and* $\{f_u^*(x)\} \rightarrow f^*(x)$, *where* $f_u^* \in X^*$, *for* f^* *in a directed*

set D. *let* U *be any neighborhood of* $f^*(x)$. *Then there exists a* $\beta \in D$ *such that*

$\beta < ʎ \Rightarrow f_ʎ^*(x) \in U$. *so* $U \cap X^* \neq \emptyset$.

Thus, $f^*(x)$ *is a limit point of* X^*.

The proof is made.

So the polars of the family of all finite subsets of X *forms a local subbase for the*

*weak** *topology* W^* *on* X^*.

*We established also that the weak** *topological space* (X^*, W^*) *is locally convex and*

Hausdorff.

2.3.3. Product Topology $\daleth^*$ *on* $\boldsymbol{X}^*$.

$Let\ X\ be\ a\ locally\ convex\ space\ over\ a\ scalar\ field\ K\ and\ X^*\ is\ dual$. Let K be equuiped with the usual topology of R.

$$let\ X^* = \underset{x\ \in\ X}{\pi} K_x = P.$$

The product copies of K indexed by $X. \Rightarrow K_x = K$.

$f^* \in X^* \equiv (f^*(x)) \in P\ i.e. the\ x - th$ coordinate of $f^*\ is\ f^*(x)$. This shows that $X^*\ is$ a product of copies of K indexed by X. Consider the polar of a point $x \in X$. $This\ is\ given\ by$

$$\{x_p\}^0 = \{f^* \in X^*: |f^*(x_p)| < 1\}$$

$$= \pi_{x_p}^{-1}\{open\ balls\ with\ radius\ 1\ centered\ at\ 0\}$$

$$= \pi_{x_p}^{-1}\{B_1(0)\} = \pi_{x_p}^{-1}\{(-1,1)\}. Open$$

$B_1(0) = \{r \in k: |r| < 1\} and\ \pi_{x_p}\ is\ the\ x_p - th\ coordinate\ projection\ from\ P\ onto\ the\ x_p - th\ coordinate.$

$Then$

$$\{x_p\}^0 = \{x_1, x_2, \dots, x_p\}^0 = \cap^p \{x_1\}^0 \cap \{x_2\}^0 \cap \dots \cap \{x_p\}^0$$

$$= \bigcap_{i=1} \pi_{x_i}^{-1}\{B_1(0)\} = \bigcap_{i=1} \pi_{x_i}^{-1}\{(-1,1)\}. Open$$

So the polars of finite subsets o$f\ X$ is a subbase for the product topology $\daleth_p\ on\ P,$ and for $P = X^* = \underset{x\ \in\ X}{\pi} K_x$,we have deduced that the topology $on\ P\ is\ the\ weak^*\ t$opology on X^*.We recall also that the product topology $\daleth_p\ is$ the point open topology or the topology of point wise convergence on X^*.

We summarize the above discussion in the proposition below;

Proposition 2.3.3

Let X be a locally convex linear topological space over $K\ and\ X^*\ its\ dual$.Then, the point open topology $\daleth_p\ on\ X^* = weak - topology\ W\ on\ X^* = product$ topology $\daleth^*\ on\ X^*\ when\ X^*\ i$ s a product space $= t$ opology of pointwise convergence $\daleth_0\ on\ X^*$.

Proof

From the proposition 4.3.1, we observed that the family polars of finite subsets of X form a subbase for topology on X^*. Thus, for $\{x_1, x_2, \dots, x_p\} \subset X$. then

$$\{x_1, x_2, \dots, x_p\}^0 = \{x_1\}^0 \cap \{x_2\}^0 \cap \dots \cap \{x_p\}^0$$

$$= \{f^* \in X^*: |f^*(x_1)| < 1\} \cap \{f^* \in X^*: |f^*(x_2)| < 1\} \cap \dots \cap \{f^* \in X^*: |f^*(x_p)| < 1\}$$

$$= \pi_{x_i}^{-1}\{f^* \in X^*: |f^*(x) < 1|\}$$

$$= \cap_{i=1}^{-1} \pi_{x_i}^{-1}\{B_1(0)\}$$

$$= \cap_{i=1}^{-1} \pi_{x_i}^{-1}\{(-1,1)\}. Open$$

Hence the point open topology is the $weak^*$ $topology$ on X^* (the topology for which a subbase at 0 are the polars of finite subsets of X.

2.3.4. Compact Open Topology Ʇ^* *on* $\boldsymbol{X}^*$.

Preamble 2.3.4.1

Let X be a locally convex linear topological space over K and X^* its $dual$. Let B denote the family of all compact subsets of X, and let B^0 denote the family of polars of sets of B. we claim that B^0 constitutes a subbase for a unique topology on X^* $called$ $Mackey$ $topology$ Ʇ_m.

Thus, the subbase for the compact open topology and that of Mackey topology are

the same.

Theorem 2.3.4

Let X be a locally convex linear topological space over K and X^* its $dual$. Then the compact open topology Ʇ_c on X^* and the Mackey topology Ʇ_m on X^* are $equivalent$.

Proof

Let B be any compact subset of X. let B^0 be the polars of sets in B denoted by

$B^0 = \{ f^* \in X^*: | f^*(x) | < 1, \forall x \in B\}$.Let $B_1(0) = \{r \in k: |r| < 1\}$. $open$ set in k.

Let U be any subbasic open set in Ʇ_c $i.e.$ $U = \{ f^* \in X^*: | f^*(x) | < 1\} \subset B_1(0), \forall x \in B\}$.

$$= \{ f^* \in X^*: f^*(B) \subset B_1(0)\}\ for\ every\ x \in B.$$

$$= \{ f^* \in X^*: | f^*(x) | < 1, \forall x \in B\}$$

$$=\ polar\ of\ compact\ subsets\ B\ of\ X.$$

$$= B^0. (a\ subbasic\ open\ sets\ in\ \text{Ʇ}_m)$$

Thus, $\text{Ʇ}_c \subseteq \text{Ʇ}_m$.

Conversely, let G be a family of compact subsets of X. we need to prove that if an arbitrary open subbasic set V in Ʇ_m , then V is also an open set inꞱ_c .

Now, G be any compact subsets of X.

let $V = \{ f^* \in X^*: | f^*(x) | < 1, \forall x \in G\}$ be a subbasic open set in Ʇ_m.

$$\Rightarrow V = \{ f^* \in X^*: | f^*(G) | < 1\} \subseteq B_1(0), \forall x \in G\}.$$

$$= \{ f^* \in X^*: | f^*(G) |\} \subseteq \{r \in k: |r| < 1\}, \forall x \in G.$$

$$= \{ f^* \in X^*: | f^*(x) | < 1\}, \forall x \in G\} =\ G^0$$

$$\Rightarrow \text{Ʇ}_m \subseteq \text{Ʇ}_c. Hence\ (\text{Ʇ}_c = \text{Ʇ}_m)$$

2.3.5. Supnorm Topology on $\boldsymbol{X}^*$. (*Strong Topology* $\text{Ʇ}_{\mathbf{0}}$ *on* $\boldsymbol{X}^*$)

Preamble 2.3.5.1

Let X be a normed linear space and X^* *its dual*. *That is* X^* *is* the linear space of all continuous functions from X *into* K, *where* K *is a scalar field*. The function $f: X \to K$ is said to be bounded if and only if

$$\|f\|_\infty = sup_{x\in X}|f(x)| < \infty.$$

Proposition 2.3.5

Let X be a normed linear space with X^* *its dual*. Then the supnorm topology on X^* is the strong topology on X^*.

Proof

Let $B_r^* = \{ f^* \in X^*: \| f^*\| < r \}$ be a given basic open set in X^* with the subnorm, then $(\frac{1}{\Lambda}) B_\Lambda^* = \{f^* \in X^*: \| f^*\| < \left(\frac{1}{\Lambda}\right) \Lambda = 1\}$.

$$\Rightarrow \Lambda B_r^* \subset B_\Lambda^* \ \dots\dots\dots\dots\dots\dots\dots\dots..(a)$$

$$\Rightarrow B_1 \ is\ bounded\ subset\ of\ X.$$

Conversely, let B be a bounded subsets in X, then $B \subseteq r\overline{B_1}$. *For some* Λ such that

$$(\Lambda B_1)^0 = \{ f^* \in X^*: \| f^*(\Lambda B_1)\| \subset \{x \in k: |x| < 1\}\}$$

$$= \{f^* \in X^*: \| f^*(B_1)\| \subset \left\{x \in k: |x| < \left|\frac{1}{\Lambda}\right|\right\}$$

$$= \{f^* \in X^*: \| f^*\| < \left|\frac{1}{\Lambda}\right|\}$$

$$= B_{\frac{1}{\Lambda}}^* \Rightarrow B_{\frac{1}{\Lambda}}^* \subseteq D^0 \dots\dots\dots\dots\dots\dots\dots\dots\dots.(b)$$

Comparing system (a) and (b), we have the conclusion that the supnorm topology and the strong topology are the same.

Remark

Since any finite subset of X is compact, the *weak* topology* w^**on* X^* or the topology of pointwise convergence, or the point open topology, or the product topology on X^* *is* weaker than the compact open topology (Mackey topology) on X^*. We mean that open sets in the *weak* topology* w^**on* X^* *are also open sets in the compact open topology on* X^*, but not conversely.

Comparing the compact open topology and the strong topology on X^*, we argued that, since every compact subset of X is obviously bounded, the compact open topology is weaker than the strong topology on X^*.

Provided X is not finitely dimensional locally convex linear topological space R^n. This is because the two topologies;- compact open topology τ_c and the strong topology$\daleth_s$ coincided when X is a finite dimensionally, locally convex linear topological space R^n.

3. Conclusion

We established that the topology on the linear topological space (X,τ) is fully determined once we know the base of neighborhood at the origin(zero), using the definition and properties of open sets as well as the continuity of the addition and scalar multiplicative operations (maps) on the space (X,τ). Also, the translation mapping

f: (X,τ) →(X,τ) is homeomorphism of (X,τ) onto (X,τ) is also established. Similarly, we established that, the only linear topological space (X,τ) which has a base u at zero such that each member v∈ u is balanced and absorbent. However, we use the definition and properties of discrete topology and that of usual topology to established that the only linear topology for a linear space X = R (or ₵) over itself is the usual topology. Furthermore, we established that a compact subset of a linear topological space (X,τ) is bounded and the mapping f: (X,τ) →(X,τ) given by f(x) = -x is continuous for every x∈X, using the property of the continuity of inverse mapping on a topological space.

Recommendation

Since the research on the Dual of linear topological space has been done by us, we therefore recommend that further research should be done on the convergence of the dual of linear topological space.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to be disclosed.

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