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Solving generalized equilibrium with iterative techniques

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Abstract

In this paper, we propose a novel iterative algorithm for approximating solutions to generalized equilibrium problems and identifying common fixed point of a finite family of nonexpansive mappings in Hilbert spaces. Under suitable control conditions on the algorithmic parameters, we establish strong convergence of the generated sequence to a unique point that simultaneously solves the generalized equilibrium problem and lies in the intersection of the fixed point sets. This limit is further characterized as the unique solution to a corresponding variational inequality. The presented results extend and improve upon several recent contributions in the literature. Moreover, both the iterative scheme and the analytic techniques employed are of independent interest and may be applicable to broader classes of problems in nonlinear analysis.

Keywords: Equilibrium Problem; Nonexpansive Mapping; Fixed Point; Hilbert Spaces; Strong Convergence; Monotone Mappings

1 Introduction

The study of iterative methods for solving nonlinear problems in Hilbert spaces has garnered significant attention due to their wide applicability in optimization, equilibrium theory, and partial differential equations. In particular, variation inequalities, fixed point problems, and generalized equilibrium problems serve as foundational models in these domains.

Let H be a Hilbert space and $C \subset H$, a nonempty, closed, and convex subset. A nonlinear operator $G: H \longrightarrow H$ is said to be k – Lipschitz if there exists constants $k, \eta > 0$ such that for all $x, y \in H$

$$\|Gx - Gy\| \leq k\|x - y\| \text{ and } \langle Gx - Gy, x - y \rangle \leq \eta\|x - y\|^2 \qquad (1)$$

When $k \in [0, 1)$, G is a contradiction; if $k = 1$, it is nonexpansive

The classical variational inequality problem seeks a point . $x^* \in C$ such that

$$\langle Gx^*, y - x^* \rangle \geq 0, \quad \forall y \in C$$

This problem is equivalent to the fixed point formulation

$$x^* = P_C(x^* - \delta Gx^*),$$

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where $\delta > 0$ is a fixed constant and P_C denotes the metric projection onto C. Under suitable conditions on G and δ, fixed point methods can be employed to approximate solutions of the variational inequality. However, the computational complexity of the projection operator P_C, especially when C has intricate geometry, has motivated the development of alternative iterative schemes.

To address this challenge, hybrid methods have been introduced. Notably, Yamada proposed a descent – type algorithm that avoids direct projection while ensuring strong convergence under monotonicity and Lipschitz conditions. This approach has inspired a range of subsequent algorithms aimed at solving variational inequalities and related problems more efficiently.

Parallel to this, equilibrium problems has emerged as a unifying frame work encompassing variational inequalities optimization problems, and Nash equilibria. Given a family of bifunctions $\{f_k\}_{k\in\Lambda} : C \times C \longrightarrow \mathbb{R}$, the generalized equilibrium problem seeks $x^* \in C$ such that

$$f_k(x^*, \quad y) \geq 0\,, \quad \forall\, y \in C, \ \forall\, k \in \Lambda.$$

When Λ is a singleton, this reduces to the classical equilibrium problem. Such models arise naturally in economics, game theory and engineering.

Recent developments have focused on finding common solutions to multiple problem classes – such as the intersection of fixed point sets of nonexpansive mappings and solution sets of equilibrium problems. Viscosity approximation methods, modified Mann iterations, and hybrid projection techniques have been employed to establish strong convergence results under various structural assumptions.

Motivated by these advances, we introduce a new iterative scheme for approximating a common solution to a finite family of nonexpansive mappings and a system of generalized equilibrium problems in Hilbert spaces. The proposed method guarantees strong convergence to a unique point that simultaneously solves the variational inequality, lies in the intersection of the fixed point sets, and satisfies the equilibrium conditions. Our results generalize and improve upon several recent contributions in the literature, and the algorithmic framework and analytical techniques developed herein are of independent interest.

2 Preliminaries

Let H be a real Hilbert space. For a sequence $\{x_n\} \subset H$, we write $x_n \longrightarrow x^*$ to denote strong convergence and $x_n \rightharpoonup x^*$ to denote weak convergence. A Banach space E is said to satisfy the *opial condition* if for every sequence $\{x_n\} \subset E$ with $x_n \rightharpoonup x^*$, the inequality

$$\liminf_{n\to\infty} \|x_n - x^*\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for all $y \in E, \ y \neq x^*$.

It is well known that Hilbert space satisfy the Opial condition (see [6]).

We shall make use of the following auxiliary results throughout the paper.

Lemma 2.1 (**Xu** [**10**]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1-\lambda_n)\, a_n + \lambda_n \sigma_n + \gamma_n, \quad n \geq 0,$$

where:

(i) $\{\lambda_n\} \subset [0\,,\ 1]$ *with* $\sum \lambda_n = \infty$;

(ii) $\liminf_{n\to\infty} \sigma_n \leq 0$;

(iii) $\gamma_n \geq 0$ *and* $\sum \gamma_n < \infty$.

Then $a_n \longrightarrow 0$ *as* $n \longrightarrow \infty$

Theorem 2.1 (**Takahashi** [7]) *Let* $C \subset H$ *be a nonempty, closed, and convex subset, and let* $f: C \times C \longrightarrow \mathbb{R}$, *be a bifunction satisfying*

$(A1)$ $f(x, x) = 0$ *for all* $x \in C$

$(A2)$ f *is monotonic* $f(x, y) + f(y, x) \leq 0$ *for all* $x, y \in C$

$(A3)$ *For all* $x, y, z \in C$, $\lim_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y)$;

$(A4)$ *For each* $x \in C$, $y \mapsto f(x, y)$ *is convex and lower continuous.*

Lemma 2.2 (**Blum and Oettli** [**2**]) *Let* $C \subset H$ *be a nonempty, closed, and convex, and let* $f: C \times C \longrightarrow \mathbb{R}$, *satisfy conditions* $(A1) - (A4)$. *Then for any* $r > 0$ *and* $x \in H$, *there exists* $z \in C$ *such that*

$$f(x, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \quad \forall y \in C.$$

Lemma 2.3 (**Combettes and Hirstoaga** [**5**]) *Let* $f: C \times C \longrightarrow \mathbb{R}$, *satisfy conditions* $(A1) - (A4)$. *Then for any* $r > 0$ *and* $x \in H$, *define the mapping* $T_r^f: H \longrightarrow C$ *by*

$$T_r^f(x) := \{ z \in C : f(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \quad \forall y \in C \}.$$

Then

(1) T_r^f *is single – valued*

(2) T_r^f *is firmly nonexpansive. That is:*

$$\left\| T_r^f x - T_r^f y \right\|^2 \leq \langle T_r^f x - T_r^f y, x - y \rangle, \ \forall x, y \in H;$$

(3) *The fixed point set* $F(T_r^f)$ *coincides with the solution set* $EP(f)$;

(4) $EP(f)$ *is closed and convex.*

Let $\{T_n\}_{n=1}^N$ *be a family of mappings on a subset* $C \subset H$ *such that* $\bigcap_{n=1}^N F(T_n) \neq 0$. *The family* $\{T_n\}$ *is said to satisfy the* $AKTT - condition$ [3] *if for bounded subset* $B \subset C$,

$$\sum_{n=1}^{N} \sup \{ \|T_{n+1}z - T_n z\| : z \in B \} < \infty.$$

Lemma 2.4 (**AKTT** [**3**]) *Let* $C \subset H$ *be a nonempty and closed, and let* $\{T_n\}$ *be a family of mappings* C *satisfying the* $AKTT - condition$. *Then for each* $x \in C$, *the sequence* $\{T_n x\}$ *converges strongly to a poit in* C *Moreover, defining*

$$Tx := \lim_{n \to \infty} T_n x, \ \forall x \in C$$

we have

$$\limsup_{n \to \infty} \sup \{ \|Tz - T_n z\| : z \in B \} = 0$$

for every bounded subset $B \subset C$ *In the sequel, we denote by* $(\{T_n\}, T)$ *a pair satisfying the* $AKTT - condition$*, with* T *defined as above and* $F := \bigcap_{n=1}^{N} Fix(T_n)$.

3 Main Results

In the analysis that follows, we adopt the assumptions $\eta < \frac{1}{2}$ in inequality (1) without loss of generality. This condition facilitates the convergence analysis and is standard in related literature. We begin by establishing a key lemma that serves as a foundation for the subsequent results.

Lemma 3.1 *Let* H *be a real Hilbert space and Let* $T: H \longrightarrow H$*, be a nonexpansive mapping . Suppose* $G: H \longrightarrow H$ *is an* $\eta -$ *strongly monotone and* $k -$*Lipschitz continuous operator. Let* $\lambda \in (0, 1)$ *and choose* $\delta \in (0, min\left\{1, \frac{2\eta}{k^2}\right\})$*. Define the mapping* $T^{\lambda}: H \longrightarrow H$ *by*

$$T^{\lambda} x = Tx - \lambda\delta G(Tx), \qquad \forall\, x \in H$$

Then T^{λ} *is a strict contraction*. Specifically, for all $x, y \in H$,

$$\left\|T^{\lambda} x - T^{\lambda} y\right\| \leq (1 - \lambda\alpha)\|x - y\|$$

where

$$\alpha := \delta(\eta - \frac{\delta k^2}{2}) \in (0, 1).$$

Proof: From $\delta < \frac{2\eta}{k^2}$, we have $0 < 2\eta - \delta k^2$. Furthermore, from $\eta < \frac{1}{2}$, we have $2\eta - \delta k^2 < 1$.

So that

$0 < 2\eta - \delta k^2 < 1$. Also, as $\delta < 1$ and $\lambda \in (0, 1)$, we obtain that $0 < \lambda\delta(2\eta - \delta k^2) < 1$.

Then for $x, y \in H$, we have

$$\begin{aligned}
\left\|T^{\lambda} x - T^{\lambda} y\right\|^2 &= \|Tx - Ty - \lambda\delta(G(Tx) - G(Ty))\|^2 \\
&= \|Tx - Ty\|^2 - 2\lambda\delta\langle G(Tx) - G(Ty), x - y\rangle + \lambda^2\delta^2\|(G(Tx) - G(Ty))\|^2 \\
&\leq \|Tx - Ty\|^2 - 2\lambda\delta\eta\|Tx - Ty\|^2 + \lambda^2\delta^2k^2\|Tx - Ty\|^2 \\
&\leq [1 - \lambda\delta(2\eta - \lambda\delta k^2)]\, \|x - y\|^2 \\
&\leq [1 - 2\lambda\delta\left(\eta - \frac{\delta k^2}{2}\right)]\, \|x - y\|^2 \\
&\leq [1 - \lambda\delta\left(\eta - \frac{\delta k^2}{2}\right)]^2 \|x - y\|^2 \\
&= [1 - \lambda\alpha]^2\, \|x - y\|^2 .
\end{aligned}$$

The proof is now complete.

Theorem 3.1 *Let* C *be a nonempty, closed, convex subset of a real Hilbert space* H*. Let* $\{f_k\}_{k=1}^{M}: C \times C \longrightarrow \mathbb{R}$*, be a family of bifunction which satisfies* $(A1) - (A4)$. Let $\{T_n\}_{n=1}^{N}$ *be a finite family of nonexpansive mappings of* C into itself and let G be a $k -$*Lipschitz* and $\eta -$ *strongly monotone mapping of* C *into* H *such that* $F := \bigcap_{n=1}^{N} Fix(T_n) \cap (\bigcap_{k=1}^{M} EP(f_k)) \neq 0$. *For an arbitrary but fixed* $\delta \in \left(0, \frac{2\eta}{k^2}\right)$*, let* $\{x_n\}$ *and* $\{y_n\}$ be sequences generated by

$$\begin{cases} x_1 \in C \\ y_n = P_C[(1-\alpha_n)x_n] \\ u_n = T_{r_M}^{f_M} T_{r_{M-1}}^{f_{M-1}} \dots T_{r_2}^{f_2} T_{r_1}^{f_1} y_n \\ z_n = \beta_n x_n + (1-\beta_n) T_n u_n \\ x_{n+1} = T^{\lambda_{n+1}} z_n = T_1 z_n - \lambda_{n+1} \delta G(T_1 z_n) \quad n \geq 1 \end{cases} \tag{2}$$

where $\{\alpha_n\}$, $\{\beta_k\}$, $\{\lambda_n\}$ are sequences in $(0\,,\ 1)$ and $r_k \in (0\,,\ \alpha_k)$ for each $k \in \{1, 2, \ \dots, \ M\}$

satisfying

$(C1)$ $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \frac{\lambda_{n+1}}{\alpha_n} = 0$;

$(C2)$ $\sum_{n=1}^{\infty} \alpha_n = \infty$

$(C3)$ $\lim_{n\to\infty} \beta_n = 0$, $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$ and $\lim_{n\to\infty} |\beta_{n+1} - \beta_n| = 0$

Suppose that $(\{f_k\}\,,\ T)$ satisfies the $AKTT - condition$. Then $\{x_n\}$ *and* $\{y_n\}$ converge strongly to an element in F, which is a unique solution of the variational inequality $VI(G\,,\ F)$

Proof: We first prove the boundedness of $\{x_n\}$. Denote $\theta^k = T_{r_M}^{f_M} T_{r_{M-1}}^{f_{M-1}} \dots T_{r_2}^{f_2} T_{r_1}^{f_1}$ for any $k \in \{1, 2, \dots, M\}$ and $\theta^0 = I$. We note that $u_n = \theta^M y_n$. From (2) we have for each $x^* \in F$ that

$$\begin{aligned} \|u_n - x^*\| &= \|\theta^M y_n - \theta^M x^*\| \\ &\leq \|y_n - x^*\| \end{aligned}$$

Also

$$\begin{aligned} \|y_n - x^*\| &= \|P_C[(1-\alpha_n)x_n] - x^*\| \\ &\leq \|(1-\alpha_n)x_n - x^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n \|x^*\| \end{aligned}$$

Thus,

$$\begin{aligned} \|z_n - x^*\| &= \|\beta_n(x_n - x^*) + (1-\beta_n)(u_n - x^*)\| \\ &\leq \beta_n \|x_n - x^*\| + (1-\beta_n)\|u_n - x^*\| \\ &\leq \beta_n \|x_n - x^*\| + (1-\beta_n)\|y_n - x^*\| \\ &\leq \beta_n \|x_n - x^*\| + (1-\beta_n)(1-\alpha_n)\|x_n - x^*\| + \alpha_n(1-\beta_n)\|x^*\| \\ &\leq 1 - \alpha_n(1-\beta_n)\|x_n - x^*\| + \alpha_n(1-\beta_n)\|x^*\| \end{aligned} \tag{3}$$

But, from (3) and Lemma 3.1, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \left\|T^{\lambda_{n+1}} z_n - x^*\right\| \\ &\leq \left\|T^{\lambda_{n+1}} z_n - T^{\lambda_{n+1}} x^*\right\| + \left\|T^{\lambda_{n+1}} x^* - x^*\right\| \\ &\leq (1 - \lambda_{n+1}\alpha)\|z_n - x^*\| + \lambda_{n+1}\delta\|G(x^*)\| \end{aligned}$$

$$\leq (1-\lambda_{n+1}\alpha)[\,(1-\alpha_n(1-\beta_n))\|x_n-x^*\| + \alpha_n(1-\beta_n)\|x^*\|\,]$$

$$+\lambda_{n+1}\alpha\frac{\delta}{\alpha}\|G(x^*)\|$$

$$\leq \max\{\max\{\|x_n-x^*\|,\ \|x^*\|\},\ \frac{\delta}{\alpha}\|G(x^*)\|\}$$

where α is as in Lemma 3.1. Hence, $\{x_n\}$ is bounded. Furthermore, $\{y_n\}$, $\{z_n\}$, $\{x_n\}$, $\{T_1 z_n\}$, $\{G(T_1 z_n)\}$, $\{x_n\}$ and $\{T_n u_n\}$ are each bounded.

We now show that $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$

Now, from (2), we have

$$\|z_n - z_{n-1}\| = \|\beta_n x_n + (1-\beta_n)T_n u_n - \beta_{n-1}x_{n-1} - (1-\beta_{n-1})T_{n-1}u_{n-1}\|$$

$$\leq \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + (1-\beta_n)\|u_n - u_{n-1}\|$$

$$+\|T_n u_{n-1} - T_{n-1}u_{n-1}\| + \beta_{n-1}\|T_{n-1}u_{n-1}\| + \beta_n\|T_n u_{n-1}\| \quad (4)$$

Since $u_n = \theta^M y_n$ and $u_{n-1} = \theta^M y_{n-1}$, then

$$\|u_n - u_{n-1}\| = \|\theta^M y_n - \theta^M y_{n-1}\|$$

$$\leq \|y_n - y_{n-1}\|$$

$$= \|P_C[(1-\alpha_n)x_n] - P_C[(1-\alpha_{n-1})x_{n-1}]\|$$

$$\leq \|(1-\alpha_n)x_n - (1-\alpha_{n-1})x_{n-1}\|$$

$$\leq \|x_n - x_{n-1}\| + \alpha_n\|x_n\| + \alpha_{n-1}\|x_{n-1}\| \quad (5)$$

Substituting (5) into (3) we have

$$\|z_n - z_{n-1}\| = \beta_n\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1}\| + (1-\beta_n)\|x_n - x_{n-1}\|$$

$$+\alpha_n\|x_n\| + \alpha_{n-1}\|x_{n-1}\| + \|T_n u_{n-1} - T_{n-1}u_{n-1}\| + \beta_{n-1}\|T_{n-1}u_{n-1}\|$$

$$+\beta_n\|T_n u_{n-1}\|$$

$$\leq \|x_n - x_{n-1}\| + (1-\beta_n)[\alpha_{n-1}\|x_{n-1}\| + \alpha_n\|x_n\|] + |\beta_n - \beta_{n-1}|\|x_{n-1}\|$$

$$+\|T_n u_{n-1} - T_{n-1}u_{n-1}\| + \beta_{n-1}\|T_{n-1}u_{n-1}\| + \beta_n\|T_n u_{n-1}\| \quad (6)$$

But

$$\|x_n - x_{n-1}\| = \left\|T^{\lambda_{n+1}}z_n - T^{\lambda_n}z_{n-1}\right\|$$

$$\leq \left\|T^{\lambda_{n+1}}z_n - T^{\lambda_{n+1}}z_{n-1}\right\| + \left\|T^{\lambda_{n+1}}z_{n-1} - T^{\lambda_n}z_{n-1}\right\|$$

$$\leq (1-\lambda_{n+1}\alpha)\|z_n - z_{n-1}\| + |\lambda_n - \lambda_{n+1}|\delta\|G(T_1 z_{n-1})\| \quad (7)$$

Therefore, from (6), we have

$$\|x_n - x_{n-1}\| \leq (1-\lambda_{n+1}\alpha)\|x_n - x_{n-1}\|$$

$$+(1-\lambda_{n+1}\alpha)[\|x_n - x_{n-1}\| + (1-\beta_n)(\alpha_{n-1}\|x_{n-1}\| + \alpha_n\|x_n\|)$$

$$+|\beta_n-\beta_{n-1}|\|x_{n-1}\|+\|T_nu_{n-1}-T_{n-1}u_{n-1}\|+\beta_{n-1}\|T_{n-1}u_{n-1}\|$$

$$+\beta_n\|T_nu_{n-1}\|]+|\lambda_n-\lambda_{n+1}|\delta\|G(T_1z_{n-1})\|$$

$$\leq(1-\lambda_{n+1}\alpha)\|x_n-x_{n-1}\|+(1-\beta_n)[\alpha_{n-1}\|x_{n-1}\|+\alpha_n\|x_n\|]$$

$$+|\beta_n-\beta_{n-1}|\|x_{n-1}\|+\|T_nu_{n-1}-T_{n-1}u_{n-1}\|+\beta_{n-1}\|T_{n-1}u_{n-1}\|$$

$$+\beta_n\|T_nu_{n-1}\|]+|\lambda_n-\lambda_{n+1}|\delta\|G(T_1z_{n-1})\|$$

$$=(1-\lambda_{n+1}\alpha)\|x_n-x_{n-1}\|+b_n$$

where

$$b_n=(1-\beta_n)[\alpha_{n-1}\|x_{n-1}\|+\alpha_n\|x_n\|]$$

$$+|\beta_n-\beta_{n-1}|\|x_{n-1}\|+\sup_{z\in u_n}\{\|T_nz-T_{n-1}z\|\}+\beta_{n-1}\|T_{n-1}u_{n-1}\|$$

$$+\beta_n\|T_nu_{n-1}\|]+|\lambda_n-\lambda_{n+1}|\delta\|G(T_1z_{n-1})\|.$$

Since $\{T_n\}$ satisfies the **AKTT** − condition and by Lemma 2.1, we have that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0 \tag{8}$$

Observe that

$$\|y_n-x_n\|=\|P_C(1-\alpha_n)x_n-P_Cx_n\|\leq\alpha_n\|x_n\|\to 0,\quad as\quad n\to\infty \tag{9}$$

We obtain for each $x^*\in F$ that

$$\|y_n-x^*\|^2=\|P_C(1-\alpha_n)x_n-x^*\|^2$$

$$\leq\|(x_n-x^*)-\alpha_nx^*\|^2$$

$$\leq[\|x_n-x^*\|+\alpha_n\|x^*\|]^2$$

$$\leq\|x_n-x^*\|^2+2\alpha_n\|x_n-x^*\|\|x^*\|+{\alpha_n}^2\|x_n\|^2$$

$$=\|x_n-x^*\|^2+\alpha_nM_1 \tag{10}$$

Where $M_1=2\|x_n-x^*\|+\alpha_n\|x^*\|$.

Since $T_{r_k}^{f_k}$ is firmly nonexpansive for each $k\in\{1,2,3,...,M\}$, so we have for each $x^*\in F$ and $k\in\{1,2,3,...,M\}$ that

$$\|\theta^ky_n-x^*\|^2=\left\|T_{r_k}^{f_k}\theta^{k-1}y_n-T_{r_k}^{f_k}\theta^{k-1}x^*\right\|^2$$

$$\leq\langle\theta^{k-1}y_n-\theta^{k-1}x^*,\ \theta^ky_n-x^*\rangle$$

$$=\frac{1}{2}(\|\theta^{k-1}y_n-x^*\|^2+\|\theta^ky_n-x^*\|^2-\|\theta^{k-1}y_n-\theta^ky_n\|^2)$$

This implies that

$$\|\theta^ky_n-x^*\|^2\ \leq\ \|\theta^{k-1}y_n-x^*\|^2-\|\theta^{k-1}y_n-\theta^ky_n\|^2 \tag{11}$$

This shows that from (10) and (11) , we have

$$\|u_n - x^*\|^2 = \|\theta^M y_n - x^*\|^2$$

$$\leq \|y_n - x^*\| - \sum_{i=1}^{M} \left\|\theta^{i-1} y_n - \theta^i y_n\right\|^2$$

$$\leq \|u_n - x^*\| - \sum_{i=1}^{M} \left\|\theta^{i-1} y_n - \theta^i y_n\right\|^2 \quad (12)$$

But

$$\|x_{n+1} - x^*\|^2 = \left\|T^{\lambda_{n+1}} z_n - x^*\right\|^2$$

$$\leq [\ \left\|T^{\lambda_{n+1}} z_n - T^{\lambda_{n+1}} x^*\right\| + \left\|T^{\lambda_{n+1}} x^* - x^*\right\|\ ]^2$$

$$\leq [\ (1 - \lambda_{n+1}\alpha)\|z_n - x^*\| + \lambda_{n+1}\delta\|G(x^*)\|\ ]^2$$

$$\leq (1 - \lambda_{n+1}\alpha)^2\|z_n - x^*\|^2 + 2\delta\lambda_{n+1}(1 - \lambda_{n+1})\|z_n - x^*\|\|G(x^*)\|$$

$$+\lambda_{n+1}^2\delta^2\|G(x^*)\|^2 \quad (13)$$

Also, from (12) we have that

$$\|z_n - x^*\|^2 = \|\beta_n(x_n - x^*) + (1 - \beta_n)T_n u_n - x^*\|^2$$

$$\leq \beta_n\|x_n - x^*\|^2 + (1 - \beta_n)\|u_n - x^*\|^2$$

$$\leq \beta_n\|x_n - x^*\|^2 + (1 - \beta_n)\left[\ \|y_n - x^*\|^2 - \sum_{i=1}^{M}\left\|\theta^{i-1} y_n - \theta^i y_n\right\|^2\ \right]$$

$$\leq \beta_n\|x_n - x^*\|^2 + (1 - \beta_n)[\ \|x_n - x^*\|^2 + \alpha_n M_1 - \sum_{i=1}^{M}\left\|\theta^{i-1} y_n - \theta^i y_n\right\|^2\ ]$$

$$= \|x_n - x^*\|^2 + \alpha_n(1 - \beta_n)M_1 - (1 - \beta_n)\sum_{i=1}^{M}\left\|\theta^{i-1} y_n - \theta^i y_n\right\|^2\ ] \quad (14)$$

Substituting (14) into (13), we haave

$$\|x_{n+1} - x^*\|^2 \leq (1 - \lambda_{n+1}\alpha)[\ \|x_n - x^*\|^2 + \alpha_n(\beta_n)M_1 - (1 - \beta_n)\sum_{i=1}^{M}\left\|\theta^{i-1} y_n - \theta^i y_n\right\|^2\ ]$$

$$+2\delta\lambda_{n+1}(1 - \lambda_{n+1}\alpha)\|z_n - x^*\|\|G(x^*)\| + \lambda_{n+1}^2\delta^2\|G(x^*)\|^2$$

$$\leq \|x_n - x^*\|^2 + \alpha_n(1 - \lambda_{n+1}\alpha)(1 - \beta_n)M_1$$

$$-(1 - \lambda_{n+1}\alpha)(1 - \beta_n)\sum_{i=1}^{M}\left\|\theta^{i-1} y_n - \theta^i y_n\right\|^2$$

$$+2\delta\lambda_{n+1}(1 - \lambda_{n+1}\alpha)\|z_n - x^*\|\|G(x^*)\| + \lambda_{n+1}^2\delta^2\|G(x^*)\|^2$$

Therefore,

$$(1-\lambda_{n+1}\alpha)(1-\beta_n)\sum_{i=1}^{M}\left\|\theta^{i-1}y_n-\theta^{i}y_n\right\|^2 \le (\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_{n+1}-x_n\|$$

$$+\alpha_n(1-\lambda_{n+1}\alpha)(1-\beta_n)M_1$$

$$+2\delta\lambda_{n+1}(1-\lambda_{n+1}\alpha)\|z_n-x^*\|\|G(x^*)\|$$

$$+\lambda_{n+1}^2\delta^2\|G(x^*)\|^2$$

Hence,

$$\lim_{n\to\infty}\sum_{i=1}^{M}\left\|\theta^{i-1}y_n-\theta^{i}y_n\right\|^2=0$$

So we can conclude that

$$\lim_{n\to\infty}\|\theta^{k-1}y_n-\theta^{k}y_n\|^2=0 \qquad (15)$$

for each $k\in\{1\,,\ 2\,,\ 3\,,\ \ldots\,,\ M\}$. Observing

$$\|u_n-y_n\|=\|\theta^{M}y_n-y_n\|$$

$$\le\|\theta^{M}y_n-\theta^{M-2}y_n\|+\|\theta^{M-2}y_n-\theta^{M-1}y_n\|+\ldots+\|\theta^{1}y_n-y_n\|$$

it follows from (15) that

$$\lim_{n\to\infty}\|u_n-y_n\|=0 \qquad (16)$$

Next we show that

$$\lim_{n\to\infty}\|T_ny_n-y_n\|=0$$

Now,

$$\|z_n-u_n\|^2=2\|z_n-x^*\|^2+2\|u_n-x^*\|^2$$

$$\le 2[\,\beta_n\|x_n-x^*\|^2+(1-\beta_n)\|T_nu_n-x^*\|^2\,]$$

$$+2\|u_n-x^*\|^2$$

$$\le 2[\,\beta_n\|x_n-x^*\|^2+(1-\beta_n)\|u_n-x^*\|^2\,]$$

$$+2\|u_n-x^*\|^2$$

$$=2\beta_n\|x_n-x^*\|^2-2\beta_n\|u_n-x^*\|^2$$

$$=2\beta_n\|x_n-u_n\|[\,\|x_n-x^*\|+\|u_n-x^*\|\,]$$

Hence,

$$\lim_{n\to\infty}\|z_n-u_n\|=0 \qquad (17)$$

Also,

$$\|u_n-x_n\|\le\|u_n-y_n\|+\|y_n-x_n\|\to 0\,,\ \text{as }\ n\to\infty$$

and

$$\|z_n - x_n\| \le \|z_n - u_n\| + \|u_n - x_n\| \to 0\,,\ \text{as}\ \ n \to \infty$$

$$\begin{aligned}\|T_n y_n - y_n\|^2 &= \|T_n u_n - x^* + \ x^* - u_n\|^2 \\ &= \|T_n u_n - x^*\|^2 + 2\langle T_n u_n - x^*,\ x^* - u_n\rangle + \|u_n - \ x^*\|^2 \\ &\le 2\|u_n - \ x^*\|^2 + 2\langle T_n u_n - u_n + u_n - x^*,\ x^* - u_n\rangle \\ &= 2\|u_n - \ x^*\|^2 + 2\langle T_n u_n - u_n\,,\ x^* - u_n\rangle - 2\|u_n - x^*\|^2 \\ &= 2\langle T_n u_n - u_n\,,\ x^* - u_n\rangle \qquad (18)\end{aligned}$$

Also from (2), we have

$$\begin{aligned}\langle z_n - \ x^*\,,\ u_n - x^*\rangle &= \ \beta_n\langle x_n - \ x^*\,,\ u_n - x^*\rangle \\ &\quad + (1-\beta_n)\langle T_n u_n - u_n + u_n - x^*,\ u_n - x^*\rangle \\ &= \ \beta_n\langle x_n - \ x^*\,,\ u_n - x^*\rangle \ + \ (1-\beta_n)\langle T_n u_n u_n\,,\ u_n - x^*\rangle \\ &\quad + (1-\beta_n)\|u_n - \ x^*\|^2\end{aligned}$$

Which implies

$$\begin{aligned}(1-\beta_n)\langle T_n u_n - u_n\,,\ x^* - u_n\rangle &= \ \beta_n\langle x_n - \ x^*\,,\ u_n - x^*\rangle - \langle z_n - \ x^*\,,\ u_n - x^*\rangle \\ &\quad + (1-\beta_n)\|u_n - \ x^*\|^2 \\ &= \ \beta_n\langle x_n - \ x^*\,,\ u_n - x^*\rangle - \langle z_n - \ x^*\,,\ u_n - x^*\rangle \\ &\quad + (1-\beta_n)\langle u_n - \ x^*\,,\ u_n - x^*\rangle \\ &= \ \beta_n\langle x_n - u_n\,,\ u_n - x^*\rangle + \langle u_n - z_n\,,\ u_n - x^*\rangle\end{aligned}$$

Using this and (18), we obtain

$$\frac{(1-\beta_n)}{2}\|T_n u_n - u_n\|^2 \le \ \beta_n\langle x_n - u_n\,,\ u_n - x^*\rangle + \langle u_n - z_n\,,\ u_n - x^*\rangle$$

Therefore,

$$\lim_{n\to\infty}\|T_n y_n - y_n\| = 0 \qquad (19)$$

Next we prove that

$$\lim_{n\to\infty}\sup\langle -G(w)\,, x_{n+1} - w\rangle \le 0 \qquad (20)$$

where w is the unique solution of variational inequality $\langle -G(w)\,, x_n - w\rangle \le 0$

$\forall\ x \in F$. Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ such that

$$\lim_{n\to\infty}\sup\langle -G(w)\,, x_n - w\rangle = \lim_{j\to\infty}\langle -G(w)\,, x_{n_j} - w\rangle$$

Since $\{x_n\}$ is bounded, without loss of generality, we assume $\{x_{n_j}\}$ itself converge weakly to a point z. Then $z \in \bigcap_{i=1}^{N} F(T_i) \cap (\bigcap_{k=1}^{M} EP(f_k))$

Since $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $u_{n_j} \rightharpoonup z$ Then, $\in \bigcap_{i=1}^{N} Fix(T_I)$, if not, i.e. $z \neq T_n z, \ \forall\, n \in \mathbb{N}$ thee fact that H is an Opial's space, for all $n \in \mathbb{N}$, we have

$$\liminf_{j\to\infty} \left\| x_{n_j} - z \right\| < \liminf_{j\to\infty} \left\| u_{n_j} - T_{n_j} z \right\|$$

$$\leq \liminf_{j\to\infty} \left\| u_{n_j} - T_{n_j} u_{n_j} \right\| + \liminf_{j\to\infty} \left\| T_{n_j} u_{n_j} - T_{n_j} z \right\|$$

$$\leq \liminf_{j\to\infty} \left\| u_{n_j} - z \right\|$$

which is a contradiction, so $z \in \bigcap_{i=1}^{N} Fix(T_i)$.

Next we show that $z \in EP(f_k)$. Note that $\theta^k y_n = T_{r_k}^{f_k} y_n$ for each $k \in \{1, 2, . . . , M\}$. Hence, for each $\eta \in C$ and $k \in \{1, 2, . . . , M\}$, we obtain

$$f_k(\theta^k y_n , \eta) + \frac{1}{r_k} \langle \eta - \theta^k y_n , \theta^k y_n - \theta^{k-1} y_n \rangle \geq 0, \ \ \forall\, \eta \in C$$

It follows from $(A2)$ that

$$\langle \eta - \theta^k y_n , \frac{\theta^k y_n - \theta^{k-1} y_n}{r_k} \rangle \geq f_k(\eta , \theta^k y_n),$$

and so

$$\langle \eta - \theta^k y_{n_j} , \frac{\theta^k y_{n_j} - \theta^{k-1} y_{n_j}}{r_k} \rangle \geq f_k\left(\eta , \theta^k y_{n_j}\right),$$

Since $\frac{\theta^k y_{n_j} - \theta^{k-1} y_{n_j}}{r_k} \to 0$, $\theta^k y_{n_j} \rightharpoonup \mathrm{z}$ and using $(A4)$ we have $f_k(\eta , \theta^k y_n) \leq 0 \ \ \forall\, \eta \in C$. For a real number t, $0 < t \leq 1$ and $\eta \in C$, let $\eta_t = t\eta + (1-t)z$. Clearly $\eta_t \in C$, so that using $(A1)$ and $(A2)$, we have

$$0 = f_k(\eta_t , \eta_t) \leq t f_k(\eta_t , \eta) + (1-t) f_k(\eta_t , z) \leq t f_k(\eta_t , \eta_t).$$

This implies $f_k(\eta_t , \eta) \geq 0$, and using this and $(A3)$ we have that $f_k(z , \eta) \geq 0 \ \forall\, \eta \in C$,

$z \in \bigcap_{i=1}^{N} Fix(T_i) \cap (\bigcap_{k=1}^{M} EP(f_k))$

Noting that z is a solution of the $VI(G , F)$, we obtain that

$$\limsup_{n\to\infty} \langle -G(w) , x_n - w \rangle = \lim_{j\to\infty} \langle -G(w) , x_{n_j} - w \rangle$$

$$= \langle -G(w) , z - w \rangle \leq 0.$$

Finally, we show that $x_n \to w$ from (2)

$$\|x_{n+1} - w\|^2 = \left\| T^{\lambda_{n+1}} z_n - w \right\|^2$$

$$= \left\| T^{\lambda_{n+1}} z_n - T^{\lambda_{n+1}} w + T^{\lambda_{n+1}} w - w \right\|^2$$

$$= \left\| T^{\lambda_{n+1}} z_n - T^{\lambda_{n+1}} w - \lambda_{n+1} \delta G(w) \right\|^2$$

$$\leq \left\| T^{\lambda_{n+1}} z_n - T^{\lambda_{n+1}} w \right\|^2 + 2\lambda_{n+1} \delta \langle -G(w) , x_{n+1} - w \rangle$$

$$\leq (1-\lambda_{n+1}\alpha)\|z_n - w\|^2 + 2\lambda_{n+1}\delta\langle -G(w)\, , x_{n+1} - w\rangle \qquad (21)$$

But

$$\begin{aligned}
\|z_n - w\|^2 &= \|\beta_n(x_n - w) + (1-\beta_n)T_n u_n - w\|^2 \\
&\leq \beta_n\|x_n - w\|^2 + (1-\beta_n)\|T_n u_n - w\|^2 \\
&\leq \beta_n\|x_n - w\|^2 + (1-\beta_n)\|x_n - w\|^2 \\
&\leq \beta_n\|x_n - w\|^2 + (1-\beta_n)[\|x_n - w\|^2 + \alpha_n M_1] \\
&= \|x_n - w\|^2 + \alpha_n M_1 \qquad (22)
\end{aligned}$$

Substituting (22) into (21), we have

$$\begin{aligned}
\|x_{n+1} - w\|^2 &\leq (1-\lambda_{n+1}\alpha)[\|x_n - w\|^2 + \alpha_n(1-\beta_n)M_1] \\
&\quad +2\lambda_{n+1}\delta\langle -G(w)\, , x_{n+1} - w\rangle \\
&= (1-\lambda_{n+1}\alpha)\|x_n - w\|^2 + (1-\lambda_{n+1}\alpha)(1-\beta_n)\alpha_n M_1 \\
&\quad +2\lambda_{n+1}\delta\langle -G(w)\, , x_{n+1} - w\rangle \\
&\leq (1-\lambda_{n+1}\alpha)\|x_n - w\|^2 + \alpha_n M_1 + 2\lambda_{n+1}\delta\langle -G(w)\, , x_{n+1} - w\rangle \\
&\leq (1-\lambda_{n+1}\alpha)\|x_n - w\|^2 + \lambda_{n+1}\alpha\left[\frac{\alpha_n}{\lambda_{n+1}\alpha}M_1 + 2\delta\langle -G(w)\, , x_{n+1} - w\rangle\right]
\end{aligned}$$

Hence, using Lemma 2.1 we have that $x_n \to w$. In order to prove the uniqueness solution of the $VI(G\, ,F)$ we assume that w^* is another solution of $VI(G\, ,F)$. Similarly, we can conclude that $\{x_n\}$ converges strongly to a point w^*. Hence, $w = w^*$, that is w is the unique solution of $VI(G\, ,F)$. This completes the proof.

4 Conclusion

In this paper, we have introduced a novel iterative algorithm for approximating a common solution to a system of generalized equilibrium problems and a finite family of nonexpansive mappings in Hilbert space. Under suitable conditions on the control parameters and operator properties, we established strong convergence of the generated sequence to a unique point that simultaneously solves the generalized equilibrium problem, lies in the intersection of the fixed point sets, and satisfies a corresponding variational inequality. The proposed method extends and improves upon several existing results in the literature, offering a unified framework that accommodates both equilibrium and fixed point formulations. Moreover, the analytical techniques employed – particularly the use of strongly monotone and Lipschitz continuous operators – may be of independent interest and applicable to broader classes of problems in nonlinear analysis and optimization.

Future research may explore extensions of this framework to Banach spaces, incorporate stochastic perturbations, or investigate accelerated schemes for improved computational efficiency.

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