(RESEARCH ARTICLE)

Demiclosedness and weak convergence of supper hybrid mappings in Banach spaces

Ikechukwu Godwin Ezugorie *

Department of Mathematics, Enugu State University of Science and Technology, Enugu, Enugu State, Nigeria.

Abstract

We introduce and study a new class of mapping in Banach Spaces, termed (α, β, γ) - supper hybrid mappings, which generalize the well – known (α, β) - generalized hybrid mappings. This extended framework encompasses a broader spectrum of nonlinear of nonlinear operators and allows for refined control via an additional parameter $\gamma \geq 0$. We establish several foundational properties of supper hybrid mappings, including quasi – nonexpansivenes and the demiclosedness principle at zero. Furthermore, we prove a nonlinear ergodic theorem of Baillon's type in Hilbert spaces for supper hybrid mappings, demonstrated weak convergence of the Cesàro means to a fixed point. Our approach leverages metric projections and techniques inspired by Takahashi, thereby extending classical fixed point theory to this new operator class.

Keywords: Supper hybrid mapping; Nonlinear ergodic theorem; Quasi – nonexpansive mapping; Fixed point; Banach space; Demiclosedness principle; Cesàro mean; Weak convergence

1. Introduction

Fixed point theory for nonlinear mappings in Banach and Hilbert spaces has seen extensive development, particularly through the study of nonexpensive, quasi – nonexpansive, hybrid mappings. Among these generalized hybrid mappings, introduced to interpolate between contractive and nonexpansive behaviours. Have proven instrumental in analyzing iterative algorithms and variational inequalities.

In this paper, we propose a new class of mappings, termed (α, β, γ)- *supper hybrid mappings*, which extend the classical (α, β) - *generalized hybrid mappings* by incorporating an additional nonnegative parameters γ. This extension allows for greater flexibility in modeling nonlinear phenomenon and unifies several operator classes under one single framework.

Our primary contributions are threefold. First, we show that supper hybrid mappings with fixed points are quasi – nonexpansive, thereby inheriting a key stability property. Second, we establish the demiclosednes principle for $(I - T)$ at zero under mild assumptions on the duality mapping, both for supper hybrid and generalized hybrid mappings. Third, we prove a nonlinear ergodic theorem of Baillon's type for supper hybrid mappings in Hilbert spaces, demonstrating weak convergence of the Cesàro means to a fixed point.

The techniques employed draw inspiration from Takahashi's work on on ergodic theorems and fixed point approximations, and our results contribute to the ongoing effort to generalize and refine convergence principle in nonlinear analysis. The structure of the paper is as follows. In section 2, we present the definition of supper hybrid mappings and establish their basic properties. Section 3 contains the main results, including the demiclosedness principle and the ergodic theorem. We conclude with remarks on potential extensions and applications.

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* Corresponding author: Ikechukwu Godwin Ezugorie

2. Preliminaries

Let E be a real Banach space with dual space E^* and let $\langle .,. \rangle$ denote the duality pairing between E and E^*. A subset $C \subset E$ is said to be convex if for all $x, y \in C$ and $\in [0,1]$, the point

$tx + (1-t)y \in C$. A mapping $J : E \longrightarrow 2^{E^*}$ is called a duality mapping if

$$J(x) := \{ x^* \in E^* : \langle x, x^* \rangle = \|x^*\|^2 = \|x\|^2 \}, \forall x \in E.$$

We say that J is *weakly* continuous if $x_n \rightharpoonup x$ in E implies $J(x_n) \longrightarrow J(x)$ in the weak topology of E^*

Let H be a real Hilbert space. The metric projection $P_C : H \longrightarrow C$ onto a nonempty closed convex subset $C \subset E$ is defined by

$$P_C x := \arg \min_{y \in C} \|x - y\|, \forall x \in H$$

It is well known that P_C is nonexpansive and satisfies the variational inequality

$$\langle x - P_C x, y - P_C x \rangle \le 0, \forall y \in C.$$

A mapping $T : C \longrightarrow C$ is called: - *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$;

quasi nonexpansive if $\|Tx - p\| \le \|x - p\|$, for all $x \in C$ and $p \in F(T)$, where

$F(T) := \{ x \in C : Tx = x \}$, denotes the set of fixed points of T.

We recall the demiclosedness principle, which plays a central role in fixed point theory:

2.1. Lemma 2.1 (Demiclosedness Principle)

Let E be a Banach space with a weakly continuous duality mapping, and let $T : C \longrightarrow E$ be a mapping. If T is quasi – nonexpansive and $x_n \rightharpoonup x$ in E with $(I - T)x_n \longrightarrow 0$, then $x \in F(T)$.

We also recall the classical Cesàro means used in ergodic theory. For a mapping $: C \longrightarrow C$, the sequence $\{ S_n x \}$ defined by

$$S_n x := \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

is called the Cesàro mean of the iterates of T. In Hilbert space, such sequence often converge weakly to a fixed point under suitable conditions.

Throughout this paper, we use the notation $T^n x$ to denote the $n-$fold composition of T applied to , and we assume that all mappings act on nonempty closed convex subsets of Banach or Hilbert spaces unless stated otherwise.

2.2. Lemma 2.2 ([2])

Assuming that E is a Banach space has a weakly continuous duality mapping with guage φ. Then for any sequences $\{x_n\}$ that converges weakly to x, we have for any $y \in E$,

$$\limsup_{n \to \infty} \Phi(\|x_n - y\|) = \limsup_{n \to \infty} \Phi(\|x_n - x\|) + \limsup_{n \to \infty} \Phi(\|x - y\|)$$

Definition 2.2 *Let K be a nonempty closed subset of a Banach space . A mapping $T : K \longrightarrow E$ is called supper hybrid if there are $\alpha, \beta, \gamma \in \mathbb{R}$ with $\gamma \ge 0$ such that for all $x, y \in K$, we have*

$$\alpha \|Tx - Ty\|^2 + (1 - \alpha + \gamma)\|x - Ty\|^2 \le (\beta + (\beta - \alpha)\gamma)\|Tx - y\|^2$$

$$+ (1 - \beta - (\beta - \alpha - 1)\gamma) \, \|x - y\|^2 + (\alpha - \beta)\gamma)\|x - Ty\|^2 + \gamma \, \|y - Ty\|^2, \dots (1)$$

We call such a mapping an (α, β, γ) - supper hybrid mapping (see [3]) . Notice that an $(\alpha, \beta, 0)$- supper hybrid mapping is (α, β)- generalized hybrid mapping, that is

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1 - \beta) \, \|x - y\|^2 \quad (2)$$

So, the class of supper hybrid mappings contains the class of generalized hybrid mappings.

3. Main Results

3.1. Proposition 3.1

Let E be a Banach space, let C be a nonempty subset of E, then a supper hybrid mappings with a fixed point is quasi – nonexpansive.

Proof: Since $T : C \longrightarrow C$ is a supper hybrid mapping for $\alpha, \beta, \gamma \in \mathbb{R}$ with $\gamma \ge 0$ and $x, y \in C$, as in (1). Let $v \in F(T)$, then we have that for any $x \in C$, from (1), that

$$\alpha\|Tx - v\|^2 \le (\beta + (\beta - \alpha)\gamma)\|Tx - v\|^2 + (1 - \beta - (\beta - \alpha - 1)\gamma)\|x - v\|^2$$

$$+(\alpha - \beta)\gamma)\|x - v\|^2 + \gamma \, \|v - v\|^2 - (1 - \alpha + \gamma)\|x - v\|^2$$

Which implies that

$$[\alpha - \beta - (\beta - \alpha)\gamma]\|Tx - v\|^2 \le [\alpha - \beta + (\alpha - \beta)\gamma]\|x - v\|^2$$

and hence $\|Tx - v\|^2 \le \|x - v\|^2$. This implies that T is quasi–nonexpansive.

3.2. Proposition 3.2

Let C be a nonempty closed convex subset of a real Banach space E, with a weakly continuous duality mapping and let $T : C \longrightarrow E$ be (α, β, γ) –supper hybrid mappings with $\alpha, \beta, \gamma \in \mathbb{R}$ with $\gamma \ge 0$. Then with $(I - T)$ is demiclosed at with 0.

Proof: Let $\{x_n\}_{n=1}^\infty$ C be a sequence in C which converges weakly to p and $\{x_n - Tx_n\}_{n=1}^\infty$

Converges strongly to 0. We show that p is a fixed point of T. Since $\{x_n\}_{n=1}^\infty$ converges weakly, it is bounded. Clearly, $\{Tx_n\}_{n=1}^\infty$ is also bounded sequence. Since $T : C \longrightarrow E$ is supper–hybrid mapping, implies that from (1), since $\{x_n\}_{n=1}^\infty$ converges weakly, it is bounded.

For each $x \in E$

Define by $f : E \longrightarrow [0, \infty)$ by

$$f(x) := \limsup_{n \to \infty} \|x_n - x\|^2$$

Then from Lemma 2.2, taking $\Phi(\|x\|) = \frac{1}{2}\|x\|^2$, we obtain,

$$f(x) = \limsup_{n \to \infty}\|x_n - p\|^2 + \|p - x\|^2, \forall x \in E$$

Thus,

$$f(x) = f(p) + \|p - x\|^2, \forall x \in E$$

and

$$f(Tp) = f(p) + \|p - Tp\|^2 \quad \dots\dots (3)$$

Observe also that from (1) and (3)

$$\alpha f(Tp) = \alpha \lim_{n\to\infty} \sup \|x_n - Tp\|^2$$

$$= \alpha \lim_{n\to\infty} \sup \|x_n - Tx_n + Tx_n - Tp\|^2$$

$$= \alpha \lim_{n\to\infty} \sup \|Tx_n - Tp\|^2$$

$$\leq \lim_{n\to\infty} \sup [(\beta + (\beta - \alpha)\gamma)\|Tx_n - p\|^2 + (1 - \beta - (\beta - \alpha - 1)) \|x_n - p\|^2$$

$$+ (\alpha - \beta)\|x_n - Tx_n\|^2 + \gamma \|p - Tp\|^2 - (1 - \alpha + \gamma)\|x_n - Tp\|^2]$$

$$= (\beta + (\beta - \alpha)\gamma) f(p) + (1 - \beta - (\beta - \alpha - 1) \gamma)f(p)$$

$$+ \gamma \|p - Tp\|^2 - (1 - \alpha + \gamma) f(Tp)$$

$$= f(p) + \gamma [f(p) + \|p - Tp\|^2] - (1 - \alpha + \gamma) f(Tp)$$

$$= f(p) + \gamma f(p) - (1 - \alpha + \gamma) f(Tp)$$

$$= f(p) - (1 - \alpha) f(Tp)$$

Therefore,

$$f(Tp) \leq f(p) \quad............ (4)$$

Hence it follows from (3) and (4) that $\|p - Tp\| = 0$.

3.3. Proposition 3.3

Let C be a nonempty closed convex subset of a real Banach space E with a weakly continuous duality mapping, and let $T : C \longrightarrow C$ be (α, β)–generalized hybrid mappings with $\alpha, \beta \in \mathbb{R}$. Then $(I - T)$ is demiclosed at 0.

Proof: From Lemma 3.2 if then we obtain the desired result. We now prove the following Nonlinear ergodic theorem of Baillon's type [1] by using the technique developed by Takahashi [4].

Theorem 3.1 *Let H be a Hilbert space and let C be closed convex subset of H, let $T : C \longrightarrow C$ be a supper hybrid mapping, with $F(T) \neq 0$ and let C be a metric projection of H onto $F(T)$. Then for $x \in C$,*

$$S_n x := \frac{1}{n} \sum_{k=0}^{n-1} T^k x ....(5)$$

converges weakly to an element p of $F(T)$, where $p = \lim_{n\to\infty} PT^n x$.

Proof: *Let $T : C \longrightarrow C$ be (α, β, γ) supper-hybrid with with $\gamma \geq 0$,* then from Proposition 3.1 T is quasi-nonexpansive, we have that $F(T)$ is closed and convex. Let $x \in C$ and let P be the metric projection of H onto $F(T)$. Then, we have

$$\|PT^n x - T^n x\| \leq \|PT^{n-1} x - T^n x\| \quad........ (6)$$

$$\leq \|PT^{n-1} x - T^{n-1} x\| \quad.......... (7)$$

This implies that $\{\|PT^n x - T^n x\|\}$ non increasing. We also know that for any $v \in C$ and $u \in F(T)$

$$\langle v - Pv , Pv - u \rangle \geq 0$$

and hence

$$\|v - Pv\|^2 \leq \langle v - Pv , Pv - u \rangle.$$

So, we get

$$\|Pv - u\|^2 = \|Pv - v + v - u\|^2$$

$$= \|Pv - v\|^2 - 2\langle Pv - v, u - v \rangle + \|v - u\|^2$$

$$= \|v - u\|^2 - \|Pv - v\|^2$$

Let $m, n \in \mathbb{N}$ with $m \geq n$. Putting $v = T^m x$ and $u = T^n x$, we have

$$\|PT^m x - PT^n x\|^2 \leq \|T^m x - PT^n x\|^2 - \|PT^m x - T^m x\|^2$$

$$\leq \|T^n x - PT^n x\|^2 - \|PT^m x - T^m x\|^2$$

So, $\{PT^n x\}$ is a Cauchy sequence. Since $F(T)$, is closed, $\{PT^n x\}$ converges strongly to an element p of $F(T)$. Then we obtain, for any $n \in \mathbb{N}$,

$$\|S_n x - u\| \leq \frac{1}{n} \sum_{k=0}^{n-1} \| T^k x - u\| \leq \|x - u\|$$

So, $\{S_n x\}$ is bounded and hence there exists a weakly convergent subsequence $\{S_{n_i} x\}$ of $S_n x\}$.

If $S_{n_i} x \rightharpoonup v$, then we have $v \in F(T)$. In fact, for any $y \in C$ and $k \in \mathbb{N} \cup \{0\}$, we have

$$0 \leq (\beta + (\beta - \alpha)\gamma)\|T^{k+1}x - y\|^2 + 1 - \beta - (\beta - \alpha - 1)\gamma)\|T^k x - y\|^2$$

$$+ (\alpha - \beta)\gamma\|T^k x - Ty\|^2 + \gamma \|y - Ty\|^2$$

$$- \alpha\|T^{k+1}x - Ty\|^2 - (1 - \alpha + \gamma)\|T^k x - Ty\|^2]$$

$$= (\beta + (\beta - \alpha)\gamma)\|T^{k+1}x - y\|^2 + (1 - \beta - (\beta - \alpha - 1)\gamma)\|T^k x - y\|^2$$

$$+ (\alpha - \beta)\gamma\|T^k x - Ty\|^2 + \gamma \|y - Ty\|^2 - \alpha[\|T^{k+1}x - y\|^2$$

$$+ \|y - Ty\|^2 + 2 \langle T^{k+1}x - y, y - Ty \rangle]$$

$$- (1 - \alpha + \gamma)[\|T^k x - y\|^2 + \|y - Ty\|^2 + 2 \langle T^k x - y, y - Ty \rangle]$$

$$= (\beta + (\beta - \alpha)\gamma)\|T^{k+1}x - y\|^2 + (1 - \beta - (\beta - \alpha - 1)\gamma)\|T^k x - y\|^2$$

$$+ (\alpha - \beta)\gamma\|T^k x - Ty\|^2 - \|y - Ty\|^2$$

$$- \alpha[\|T^{k+1}x - y\|^2 + 2 \langle T^{k+1}x - y, y - Ty \rangle] - (1 - \alpha + \gamma)[\|T^k x - y\|^2$$

$$+ 2 \langle T^k x - y, y - Ty \rangle] \quad \dots\dots (8)$$

Summing up the inequality (8) with respect to $k = 0, 1, 2, 3, \ldots, n - 1$, we get

$$0 \leq (\beta + (\beta - \alpha)\gamma)\|T^n x - y\|^2 + (1 - \beta - (\beta - \alpha - 1)\gamma)\|x - y\|^2$$

$$+ (\alpha - \beta)\gamma\|x - Ty\|^2 - n \|y - Ty\|^2 - \alpha[\|T^n x - y\|^2$$

$$+ 2 \langle (n + 1)S_{(n+1)}x - x - ny, y - Ty \rangle]$$

$$- (1 - \alpha + \gamma)[\|x - y\|^2 + 2 \langle x - y, y - Ty \rangle] \quad \dots\dots\dots (9)$$

Dividing the inequality (9) by n, we have

$$0 \leq \frac{(\beta + (\beta - \alpha)\gamma)}{n}\|T^n x - y\|^2 + \frac{(1 - \beta - (\beta - \alpha - 1)\gamma)}{n}\|x - y\|^2$$

$$+ (\alpha - \beta)\frac{\gamma}{n}\|x - Ty\|^2 - \|y - Ty\|^2 - \alpha[\frac{1}{n}\|T^n x - y\|^2 \quad \text{.........} \quad (10)$$

$$+ 2 \langle \frac{(n+1)}{n} S_{(n+1)}x - \frac{x}{n} - y , y - Ty \rangle]$$

$$- (1 - \alpha + \gamma)[\frac{1}{n}\|x - y\|^2 + \frac{2}{n} \langle x - y , y - Ty \rangle], \quad \text{............} \quad (11)$$

where $\sum_{k=0}^{n} T^{k+1} x = (n+1)S_{(n+1)}x - x$ from (5). Replacing n by n_i and letting by $n_i \to \infty$, we obtain from $S_{(n_i+1)}x \rightharpoonup v$ that

$$0 \leq - \|y - Ty\|^2 \quad \text{............} \quad (12)$$

Putting $y = v$ in (12) we get

$$0 \leq - \|v - Tv\|^2,$$

that is

$$\|v - Tv\|^2 \leq 0$$

Hence, $Tv = v$. To complete the proof, it is sufficient to show that if $S_{(n_i+1)}x \rightharpoonup v$ then $v = p$. We have that

$$\langle T^k x - PT^k x , PT^k x - u \rangle \geq 0$$

for all $u \in F(T)$. Since $\{\|T^k x - PT^k x\|\}$ is nonincreasing, we have

$$\langle u - p , T^k x - PT^k x \rangle \leq \langle PT^k x - p , T^k x - PT^k x \rangle$$

$$\leq \|PT^k x - p\| . \|T^k x - PT^k x\|$$

$$\leq \|PT^k x - p\| . \|x - Px\|$$

Adding these inequalities from $k = 0$ to $k = n - 1$ and dividing by n, we have

$$\langle u - p , S_n x - \frac{1}{n}\sum_{k=0}^{n-1} PT^k x \rangle \leq \frac{\|x - Px\|}{n}\sum_{k=0}^{n-1}\|PT^k x - p\|.$$

Since, $S_{(n_i+1)}x \rightharpoonup v$ and $PT^k x \to p$, we have

$$\langle u - p , v - p \rangle \leq 0.$$

We know $v \in F(T)$. So, putting $u = v$, we have $\langle v - p , v - p \rangle \leq 0$ and hence $\|v - p\|^2 \leq 0$. So, we obtain $v = p$. This completes the proof.

Corollary 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H, and let $T : C \rightarrow C$ be a supper hybrid mapping, with nonempty fixed point set $F(T)$. Then, for any $x \in C$, the Cesaro means*

$$S_n x := \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to a point $p \in F(T)$.

Proof: This follows directly from the nonlinear ergodic theorem established in Theorem 3.1, together with the demiclosedness principle and the weak compactness of closed convex subsets in Hilbert spaces.

Applications

Variational Inequalities: The convergence of Cesaro means for supper hybrid mappings can be used to approximate solutions of variational inequality problems of the form: find $x^* \in C$ such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \forall y \in C,$$

where $A: C \longrightarrow H$ is a monotone operator. By constructing suitable supper hybrid mappings associated with the resolvent of A, one can apply the ergodic theorem to obtain weak convergence to a solution. is

Convex Feasibility Problems: In the context of finding a point in the intersection of convex sets $C_1, C_2, C_3, \ldots, C_m \subset H$ supper hybrid mapping can be designe to encode projection – based iterative schemes. The ergodic convergence of Cesàro means then provides a mechanism for approximating feasible points when direct projection is computationally expensive or infeasible.

4. Conclusion

In this paper, we have introduced and analyzed a nonlinear ergodic theorem for a new class of mappings termed supper hybrid mappings in Banach and Hilbert spaces. By employing the demiclosedness principle and properties of quasi-nonexpansive mappings, we established the weak convergence of Cesàro means to fixed points under mild assumptions. A key corollary demonstrates that such convergence holds for any initial point in the domain, thereby extending classical ergodic results to a broader class of nonlinear operators.

Beyond its theoretical significance, the main result admits applications to variational inequality problems and convex feasibility formulations, where supper hybrid mappings can be used to constructive iterative schemes with guaranteed convergence. These findings offer a unified framework for analyzing nonlinear iterative process in infinite-dimensional settings.

Further research may focus on quantitative convergence rates, stability under perturbations, and extensions to more general classes of mappings. Applications to monotone inclusion problems and operator splitting methods also present promising directions.

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