World Journal of Advanced Research and Reviews
eISSN: 2581-9615 CODEN (USA): WJARAI
Cross Ref DOI: 10.30574/wjarr
Journal homepage: https://wjarr.com/

(RESEARCH ARTICLE)

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Generalized Technique for constructing the closed form of the generating functions of certain classes of sequences

Chika Moore [1], Justina Ebele Okeke [2,*] and Elias Ikechukwu Chukwuma [2]

[1] *Department of Mathematics, Faculty of Physical Sciences, Nnamdi Azikiwe University, Awka, Anambra State, Nigeria.*
[2] *Department of Mathematics, Faculty of Physical Sciences, Chukwuemeka Odumegwu Ojukwu University, Anambra State, Nigeria.*

World Journal of Advanced Research and Reviews, 2024, 23(01), 1046–1064

Publication history: Received on 28 May 2024; revised on 10 July 2024; accepted on 12 July 2024

Article DOI: https://doi.org/10.30574/wjarr.2024.23.1.2077

Abstract

We develop generalized techniques for constructing the closed form of the generating functions of sequences of the form $f_{m+1} = k(f_m + m^k)\ for\ all\ m \geq 0, k > 0\ and\ f_0 = 0\ and\ \ f_{m+1} = k(f_m + (m-1)^k) for\ all\ m \geq 1, k > 0\ and\ f_1 = 0$ using method of differencing in generating function. Also, the techniques of generating functions were applied to solve some important problems of recurrence relations. The findings of this study provide generalized technique and fast method of obtaining the closed form of the generating functions of sequences of the form $f_{m+1} = k(f_m + m^k)\ for\ all\ m \geq 0, k > 0\ and\ f_0 = 0$ and $f_{m+1} = k(f_m + (m-1)^k)\ for\ all\ m \geq 1, k > 0\ and\ f_1 = 0$.

Keywords: Generating function; Recursive sequence; Closed form; Sequence of numbers and differencing in generating function

1. Introduction

Generating functions transform an infinite sequence into the coefficients of power series. It enables us to approach problem of sequence using the methods we have for algebraic problems. Given any power series, we treat this power series as an algebraic object where the convergence of the series is considered a non-issue. Generating functions was introduced by Abraham de Moivre in 1730 where it was used to solve problems of linear recurrence [1]. Generating functions provide a bridge between continuous and discrete mathematics. It plays an important role in the analysis of discrete problems in mathematics with function. Its beauty can be appreciated because it can be used to manipulate many complicated mathematical problems with very little effort as question about convergence is not required. The major idea of a generating function is to use a single function instead of an infinite sequence of numbers. Generating functions have proved to be very useful tools which facilitate the solution of various classes of counting problems as questions about the convergence are not required. Thus, this enables the solutions of difficult varieties of counting problems with little effort [2].

There are countless types of generating functions which includes Dirichlet series, exponential generating functions, Bell Series, ordinary generating functions and Lambert series. Every sequence has one generating function or the other but the way they are handled varies except Dirichlet series and Lambert series which requires its indices to start with one rather than zero. Generating functions are used to find exact formula of a sequence, find recurrence formulas, find statistical properties of a sequence, prove identities and a lot of others [3]. Sometimes, generating functions can be expressed in a closed form instead of as a series [4].

* Corresponding author: Justina Ebele Okeke

In this study, the generalized technique for constructing the closed form of the generating functions of the sequences of the form $f_{m+1} = k(f_m + m^k)\ for\ all\ m \geq 0, k > 0\ and\ f_0 = 0\ and\ f_{m+1} = k(f_m + (m-1)^k) for\ all\ m \geq 1, k > 0\ and\ f_1 = 0\ were\ developed$. Furthermore, the techniques of generating functions were applied to solve some important problems of recurrence relations.

2. Basic Definitions

2.1. Generating Function

It is a formal power series of form

$$B(y) = \sum_{n=0}^{\infty} a_n y^n = a_0 + a_1 y + a_2 y^2 + \cdots \ (2.1)$$

Where a_n is the coefficient of the series [5], [6].

2.2. Differencing in Generating Function

The technique of differencing in generating function is, when an infinite number of sequences are given, it helps to search for a single function which encodes the infinite number of the sequence and it also help to determine the closed form expressions for the generating functions. For instance; consider the sequence 1, 3, 5, 7, 9, ..., we can apply the technique of differencing in generating function to find the closed form of the generating functions as shown below;

$$Let\ g(y) = 1 + 3y + 5y^2 + 7y^3 + 9y^4 + \cdots$$

$$-\,yg(y) = 0 + y + 3y^2 + 5y^3 + 7y^4 + 9y^5 + \cdots$$

$$g(y)(1-y) = 1 + 2y + 2y^2 + 2y^3 + 2y^4 + \cdots$$

$$g(y)(1-y) = 1 + \left(\frac{2y}{1-y}\right)$$

$$g(y) = \frac{1+y}{(1-y)^2}\ [7]$$

2.3. Sequence and Series

A sequence of numbers is the orderly presentation of numbers

$$a_1, a_2, a_3, \dots\ , a_n, \dots$$

where a_1 is the first term, a_2 the second term, a_3 the third term, … , a_n the nth term, …. Occasionally, there is a recursive relationship between a term and the preceding or succeeding terms. Then, a series on the other hand is then the infinite sum of the terms of a sequence [8]

2.4. Closed Form

In building generating functions, it is motivating to check if the generating function has a closed form expression. That is an expression that represents the coefficient of the general term. For example, the expression

$$g(y) = \sum_{n\geq 0} f^n y^n\ \ is\ not\ in\ a\ closed\ form\ whereas,$$

$g(y) = \frac{1}{1-fy}$, a similar expression to some extent, is a closed form [9

2.5. Encoding

In the context of generating functions, encoding refers to the process of representing sequence of numbers or objects as a single function typically a formal power series [9]

2.6. Combinatorics

It is a mathematical field that deals with the problems of arrangement and selection together with their operations in a finite set which has some constraints [10], [11]

2.7. Fibonacci Sequence

It is a sequence that has the recurrence relation

$$a_{m+1} = a_m + a_{m-1} \; with \; a_{-1} = 0, a_0 = 1 \; and \; a_1 = 1$$

The general Fibonacci sequence is 0, 1, 1, 2, 3, 5, 8, 13, … [6]

2.8. Recurrence Relation

It is an equation which expresses each element of a sequence as a function of the preceding one. Example of recurrence relation is the Fibonacci sequence [6]

2.9. Exponential Generating Function

Exponential generating function of the sequence $\{a_n\}_{n\geq 0}$ is any formal power series that is of the form

$$B(y) = \sum_{n\geq 0} a_n \frac{y^n}{n!}$$

Where $B(y)$ is the exponential generating function of $a_0, a_1, a_2, a_3, a_{4,}$ … [6]

2.10. Convergent Series

$$Let \; \mathcal{s} = \sum_{k\geq 0} f_k \; be \; a \; series.$$

The sequence $\{S_n\}$ *given by*

$$S_n = \sum_{k=0}^{n} f_n$$

is called the sequence of partial sums of the series. A series $\mathcal{s}$ converges if and only if the sequence $\{S_n\}$ of partial sums of the series converges. In other words, a series converges if the limit of the series exists [12], [6]

3. Important Theorems

3.1. Taylor Series Theorem

The Taylor series of complex or real valued function $f(y)$ which is infinitely differentiable at complex or real number a is the power series

$$f(y) = f(a) + \frac{f'(a)}{1!}(y-a) + \frac{f''(a)}{2!}(y-a)^2 + \frac{f'''(a)}{3!}(y-a)^3 + \cdots = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(y-a)^n$$

Where $f^n(a)$ is the nth derivative of $f \; at \; y = a$. When $a = 0$ in the Taylor series, the series can also be called Maclaurin series [6]

3.2. Newton's Binomial Theorem

The binomial theorem deals with the expansion of any expression of this kind $(m+n)^p$. It states that for any non-negative integer p, we have

$$(m+n)^p = \sum_{r=0}^{p} \binom{p}{r} m^r n^{p-r}$$

Where

$$\binom{p}{r} = \frac{p!}{(p-r)!\,r!}$$

is the coefficient of the binomials [13], [6], [10].

3.3. Proposition

Let $\{f_n\}$ be a Fibonacci sequence and let F be a generating function given by

$$F(x) = \sum_{n\geq 0} f_n \frac{y^n}{n!} = \sum_{n\geq 0} \frac{f_n}{n!} y^n$$

Then F satisfies the second order ordinary differential equation given by

$$F'' - F' = F$$

Proof

$$F'^{(y)} = \frac{d}{dy}\sum_{n\geq 0} f_n \frac{y^n}{n!} = \sum_{n\geq 0} \frac{f_n}{n!}\frac{d}{dy}(y^n) = \sum_{n\geq 0} \frac{nf_n}{n!} y^{n-1}$$

$$= \sum_{n\geq 1} \frac{f_n}{(n-1)!} y^{n-1} = \sum_{n\geq 0} \frac{f_{n+1}}{n!} y^n = \sum_{n\geq 0} f_{n+1} \frac{y^n}{n!}$$

$$F''(y) = \frac{d^2}{dy^2}\left(\sum_{n\geq 0} f_n \frac{y^n}{n!}\right) = \sum_{n\geq 0} f_n \frac{d^2}{dy^2}\left(\frac{y^n}{n!}\right)$$

$$= \sum_{n\geq 0} f_n \frac{n(n-1)}{n!} y^{n-2} = \sum_{n\geq 0} f_n \frac{y^{n-2}}{(n-2)!}$$

$$= \sum_{n\geq 0} f_{n+2} \frac{y^n}{n!} = \sum_{n\geq 0} (f_{n+1} + f_n)\frac{y^n}{n!} = \sum_{n\geq 0} f_{n+1} \frac{y^n}{n!} + \sum_{n\geq 0} y_n \frac{y^n}{n!}$$

$$F''(y) = F'(y) + F(y)$$

So that $F'' - F' = F\ proved$.

3.4. Euler's Number Triangle

In 1755, Leonhard Euler developed numbers called the Euler's numbers. When the Euler's numbers are arranged in triangular form, it is called the Eulerian triangle numbers. It is the coefficient of the Euler's polynomials and its formula is

$$C(k,n) = \left\langle {k \atop n} \right\rangle = \sum_{i=0}^{n} (-1)^i \binom{k+1}{i} (n+1-i)^k \ (3.1)$$

For instance,

$$C(4,3) = \sum_{i=0}^{3} (-1)^i \binom{4+1}{i} (3+1-i)^4$$

$$= (-1)^0 \binom{5}{0} (3+1-0)^4 + (-1)^1 \binom{5}{1} (3+1-1)^4 + (-1)^2 \binom{5}{2} (3+1-2)^4 + (-1)^3 \binom{5}{3} (3+1-3)^4$$

$$= (1)(1)(4)^4 + (-1)(5)(3)^4 + (1)(10)(2)^4 + (-1)(10)(1)^4$$

$$= 256 - 405 + 160 - 10 = 416 - 415 = 1$$

$$\therefore C(4,3) = 1$$

The polynomials of Euler popularly called Eulerian polynomials can be presented in the form of generating function of powers

$$g(y) = \frac{yC_d(y)}{(1-y)^{d+1}}$$

$$Where\ C_d(y) = \sum_{k=0}^{d} C(d,k) y^k\ for\ d \geq 0$$

Thus, the Euler's number triangle is shown in the table below

$k \setminus n$	0	1	2	3	4	5	6	7	8
0	1								
1	1								
2	1	1							
3	1	4	1						
4	1	11	11	1					
5	1	26	66	26	1				
6	1	57	302	302	57	1			
7	1	120	1191	2416	1191	120	1		
8	1	247	4293	15619	15619	4293	247	1	
9	1	502	14608	88234	156190	88234	14608	502	1

The recursion rule is stated in (3.1) above and $C(0,0) = 1$. The cells that are empty outside the triangle are taken to be zero value [9].

4. Research Results

4.1. Closed form of the generating functions of sequences of the form $\boldsymbol{f_{m+1} = k(f_m + m^k)\ for\ all\ m \geq 0, k > 0\ and\ f_0 = 0}$

To develop the generalized technique for constructing the closed form of the generating functions of sequences of the form

$$f_{m+1} = k(f_m + m^k)\ for\ all\ m \geq 0, k > 0\ and\ f_0 = 0,$$

we consider the following step by step approach;

1. $\boldsymbol{When\ k = 1}, we\ have\ f_{m+1} = 1(f_m + m^1) = (f_m + m)\ for\ all\ m \geq 0\ and\ f_0 = 0$

$$The\ sequence\ of\ f_{m+1} = (f_m + m) = (0,1,3,6,10,15,\dots)$$

Thus, the generating function of the sequence

$$f_{m+1} = (f_m + m) = (0,1,3,6,10,15,\dots)\ is$$

$$g_1(y) = \sum_{m=0}^{\infty} f_{m+1} y^m = \sum_{m=0}^{\infty} (0,1,3,6,10,15,\dots) y^m$$

$$g_1(y) = y + 3y^2 + 6y^3 + 10y^4 + 15y^5 + \cdots$$

Applying the technique of differencing in generating function to get its closed form, we have

$$g_1(y) = y + 3y^2 + 6y^3 + 10y^4 + 15y^5 + \cdots$$

$$-\ yg_1(y) = y^2 + 3y^3 + 6y^4 + 10y^5 + \cdots$$

$$g_1(y)[1 - y] = y + 2y^2 + 3y^3 + 4y^4 + 5y^5 + \cdots\ (4.1)$$

$$Let\ A = y + 2y^2 + 3y^3 + 4y^4 + 5y^5 + \cdots$$

$$-\ yA = y^2 + 2y^3 + 3y^4 + 4y^5 + \cdots$$

$$A[1 - y] = y + y^2 + y^3 + y^4 + y^5 + \cdots$$

$$A[1 - y] = \frac{y}{(1 - y)}$$

$$\therefore A = \frac{y}{(1 - y)^2}$$

$$Substitute\ A = \frac{y}{(1 - y)^2}\ in\ (4.1), we\ have$$

$$g_1(y)[1 - y] = A = \frac{y}{(1 - y)^2}$$

$$\therefore g_1(y) = \frac{y}{(1 - y)^3}$$

Therefore, the closed form of the generating function for

$$f_{m+1} = 1(f_m + m^1) = (f_m + m)\ for\ all\ m \geq 0\ and\ f_0 = 0\ is$$

$$g_1(y) = \frac{y}{(1 - y)^2} = \frac{yc_1(y)}{(1 - y)(1 - y)^2}$$

2. $\boldsymbol{When\ k = 2}, we\ have\ f_{m+1} = 2(f_m + m^2)\ for\ all\ m \geq 0\ and\ f_0 = 0$

$$The\ sequence\ of\ f_{m+1} = 2(f_m + m^2) = (0,2,12,42,116,\dots)$$

Thus, the generating function of the sequence

$$f_{m+1} = 2(f_m + m^2) = (0,2,12,42,116,\dots)\ is$$

$$g_2(y) = \sum_{m=0}^{\infty} f_{m+1} y^m = \sum_{m=0}^{\infty} (0,2,12,42,116, \dots) y^m$$

$$g_2(y) = 2y + 12y^2 + 42y^3 + 116y^4 + \cdots$$

Applying the technique of differencing in generating function to get its closed form, we have

$$g_2(y) = 2y + 12y^2 + 42y^3 + 116y^4 + \cdots$$

$$-yg_2(y) = 2y^2 + 12y^3 + 42y^4 + \cdots$$

$$g_2(y)[1-y] = 2y + 10y^2 + 30y^3 + 74y^4 + \cdots$$

$$g_2(y)[1-y] = 2(y + 5y^2 + 15y^3 + 37y^4 + \cdots) \ (4.2)$$

$$Let\ A = y + 5y^2 + 15y^3 + 37y^4 + \cdots$$

$$-yA = y^2 + 5y^3 + 15y^4 + \cdots$$

$$A[1-y] = y + 4y^2 + 10y^3 + 22y^4 + \cdots \ (4.3)$$

$$Let\ B = y + 4y^2 + 10y^3 + 22y^4 + \cdots$$

$$-yB = y^2 + 4y^3 + 10y^4 + \cdots$$

$$B[1-y] = y + 3y^2 + 6y^3 + 12y^4 + \cdots$$

$$B[1-y] = y + \frac{3y^2}{(1-2y)} = \frac{y(1-2y) + 3y^2}{(1-2y)} = \frac{y - 2y^2 + 3y^2}{(1-2y)} = \frac{y^2 + y}{(1-2y)}$$

$$\therefore B = \frac{y^2 + y}{(1-2y)(1-y)}$$

$$Substitute\ B = \frac{y^2 + y}{(1-2y)(1-y)}\ in\ (4.3), we\ have$$

$$A[1-y] = \frac{y^2 + y}{(1-2y)(1-y)}$$

$$\therefore A = \frac{y^2 + y}{(1-2y)(1-y)^2}$$

$$Substitue\ A = \frac{y^2 + y}{(1-2y)(1-y)^2}\ in\ (4.2), we\ have$$

$$g_2(y)[1-y] = 2\left(\frac{y^2 + y}{(1-2y)(1-y)^2}\right) = \frac{2(y^2 + y)}{(1-2y)(1-y)^2} = \frac{2y(y+1)}{(1-2y)(1-y)^2}$$

$$\therefore g_2(y) = \frac{2y(y+1)}{(1-2y)(1-y)^3}$$

Therefore, the closed form of the generating function for

$$f_{m+1} = 2(f_m + m^2)\ for\ all\ m \geq 0\ and\ f_0 = 0\ is$$

$$g_2(y) = \frac{2y(y+1)}{(1-2y)(1-y)^3} = \frac{2yc_2(y)}{(1-2y)(1-y)^3}$$

3. $\boldsymbol{When\ k = 3}, we\ have\ f_{m+1} = 3(f_m + m^3)\ for\ all\ m \geq 0\ and\ f_0 = 0$

$$The\ sequence\ of\ f_{m+1} = 3(f_m + m^3) = (0, 3,33,180,732,2571, \dots)$$

Therefore, the generating function of the sequence

$$f_{m+1} = 2(f_m + m^3) = (0, 3,33,180,732,2571, \dots)\ is$$

$$g_3(y) = \sum_{m=0}^{\infty} f_{m+1} y^m = \sum_{m=0}^{\infty} (0, 3,33,180,732,2571, \dots) y^m$$

$$g_3(y) = 3y + 33y^2 + 180y^3 + 732y^4 + 2571y^5 + \cdots$$

Applying the technique of differencing in generating function to get its closed form, we have

$$g_3(y) = 3y + 33y^2 + 180y^3 + 732y^4 + 2571y^5 + \cdots$$

$$-yg_3(y) = 3y^2 + 33y^3 + 180y^4 + 732y^5 + \cdots$$

$$g_3(y)[1-y] = 3y + 30y^2 + 147y^3 + 552y^4 + 1839y^5 + \cdots$$

$$g_3(y)[1-y] = 3(y + 10y^2 + 49y^3 + 184y^4 + 613y^5 + \cdots) \quad (4.4)$$

$$Let\ A = y + 10y^2 + 49y^3 + 184y^4 + 613y^5 + \cdots$$

$$-yA = y^2 + 10y^3 + 49y^4 + 184y^5 + \cdots$$

$$A[1-y] = y + 9y^2 + 39y^3 + 135y^4 + 429y^5 + \cdots \quad (4.5)$$

$$Let\ B = y + 9y^2 + 39y^3 + 135y^4 + 429y^5 + \cdots$$

$$-yB = y^2 + 9y^3 + 39y^4 + 135y^5 + \cdots$$

$$B[1-y] = y + 8y^2 + 30y^3 + 96y^4 + 294y^5 + \cdots \quad (4.6)$$

$$Let\ C = y + 8y^2 + 30y^3 + 96y^4 + 294y^5 + \cdots$$

$$-yC = y^2 + 8y^3 + 30y^4 + 96y^5 + \cdots$$

$$C[1-y] = y + 7y^2 + 22y^3 + 66y^4 + 198y^5 + \cdots$$

$$C[1-y] = y + 7y^2 + \frac{22y^3}{(1-3y)} = \frac{(y+7y^2)(1-3y) + 22y^3}{(1-3y)}$$

$$C[1-y] = \frac{y - 3y^2 + 7y^2 - 21y^3 + 22y^3}{(1-3y)} = \frac{y^3 + 4y^2 + y}{(1-3y)}$$

$$\therefore C = \frac{y^3 + 4y^2 + y}{(1-3y)(1-y)}$$

$$Substitute\ C = \frac{y^3 + 4y^2 + y}{(1-3y)(1-y)}\ in\ (4.6), we\ have$$

$$B[1-y] = \frac{y^3+4y^2+y}{(1-3y)(1-y)}$$

$$\therefore B = \frac{y^3+4y^2+y}{(1-3y)(1-y)^2}$$

$$Substitute\ B = \frac{y^3+4y^2+y}{(1-3y)(1-y)^2}\ in\ (4.5), we\ have$$

$$A[1-y] = \frac{y^3+4y^2+y}{(1-3y)(1-y)^2}$$

$$\therefore A = \frac{y^3+4y^2+y}{(1-3y)(1-y)^3}$$

$$Substitute\ A = \frac{y^3+4y^2+y}{(1-3y)(1-y)^3}\ in\ (4.4), we\ have$$

$$g_3(y)[1-y] = 3\left(\frac{y^3+4y^2+y}{(1-3y)(1-y)^3}\right) = \frac{3(y^3+4y^2+y)}{(1-3y)(1-y)^3} = \frac{3y(y^2+4y+1)}{(1-3y)(1-y)^3}$$

$$\therefore g_3(y) = \frac{3y(y^2+4y+1)}{(1-3y)(1-y)^4}$$

Therefore, the closed form of the generating function for

$$f_{m+1} = 3(f_m + m^3)\ for\ all\ m \geq 0\ and\ f_0 = 0\ is$$

$$g_3(y) = \frac{3y(y^2+4y+1)}{(1-3y)(1-y)^4} = \frac{3yc_3(y)}{(1-3y)(1-y)^4}$$

4. $\boldsymbol{When\ k = 4}, we\ have\ f_{m+1} = 4(f_m + m^4)\ for\ all\ m \geq 0\ and\ f_0 = 0$

The sequence of

$$f_{m+1} = 4(f_m + m^4) = (0, 4,80,644,3600,16900,72784, \dots)$$

Thus, the generating function of the sequence

$$f_{m+1} = 4(f_m + m^4) = (0, 4,80,644,3600,16900,72784, \dots)\ is$$

$$g_4(y) = \sum_{m=0}^{\infty} f_{m+1}y^m = \sum_{m=0}^{\infty}(0, 4,80,644,3600,16900,72784, \dots)y^m$$

$$g_4(y) = 4y + 80y^2 + 644y^3 + 3600y^4 + 16900y^5 + 72784y^6 + \cdots$$

Applying technique of differencing in generating function to get its closed form, we have

$$g_4(y) = 4y + 80y^2 + 644y^3 + 3600y^4 + 16900y^5 + 72784y^6 + \cdots$$

$$-\,yg_4(y) = 4y^2 + 80y^3 + 644y^4 + 3600y^5 + 16900y^6 + \cdots$$

$$g_4(y)[1-y] = 4y + 76y^2 + 564y^3 + 2956y^4 + 13300y^5 + 55884y^6 + \cdots$$

$$g_4(y)[1-y] = 4(y + 19y^2 + 141y^3 + 739y^4 + 3325y^5 + 13971y^6 + \cdots)\ (4.7)$$

$$Let\ A = y + 19y^2 + 141y^3 + 739y^4 + 3325y^5 + 13971y^6 + \cdots$$

$$-\ yA = y^2 + 19y^3 + 141y^4 + 739y^5 + 3325y^6 + \cdots$$

$$A[1 - y] = y + 18y^2 + 122y^3 + 598y^4 + 2586y^5 + 10646y^6 + \cdots \quad (4.8)$$

$$Let\ B = y + 18y^2 + 122y^3 + 598y^4 + 2586y^5 + 10646y^6 + \cdots$$

$$-\ yB = y^2 + 18y^3 + 122y^4 + 598y^5 + 2586y^6 + \cdots$$

$$B[1 - y] = y + 17y^2 + 104y^3 + 476y^4 + 1988y^5 + 8060y^6 + \cdots \quad (4.9)$$

$$Let\ C = y + 17y^2 + 104y^3 + 476y^4 + 1988y^5 + 8060y^6 + \cdots$$

$$-\ yC = y^2 + 17y^3 + 104y^4 + 476y^5 + 1988y^6 + \cdots$$

$$C[1 - y] = y + 16y^2 + 87y^3 + 372y^4 + 1512y^5 + 6072y^6 + \cdots \quad (4.10)$$

$$Let\ D = y + 16y^2 + 87y^3 + 372y^4 + 1512y^5 + 6072y^6 + \cdots$$

$$-\ yD = y^2 + 16y^3 + 87y^4 + 372y^5 + 1512y^6 + \cdots$$

$$D[1 - y] = y + 15y^2 + 71y^3 + 285y^4 + 1140y^5 + 4560y^6 + \cdots$$

$$D[1 - y] = y + 15y^2 + 71y^3 + \frac{285y^4}{(1 - 4y)} = \frac{(y + 15y^2 + 71y^3)(1 - 4y) + 285y^4}{(1 - 4y)}$$

$$D[1 - y] = \frac{y - 4y^2 + 15y^2 - 60y^3 + 71y^3 - 284y^4 + 285y^4}{(1 - 4y)} = \frac{y^4 + 11y^3 + 11y^2 + y}{(1 - 4y)}$$

$$\therefore D = \frac{y^4 + 11y^3 + 11y^2 + y}{(1 - 4y)(1 - y)}$$

$$Substitute\ D = \frac{y^4 + 11y^3 + 11y^2 + y}{(1 - 4y)(1 - y)}\ in\ (4.10), we\ have$$

$$C[1 - y] = \frac{y^4 + 11y^3 + 11y^2 + y}{(1 - 4y)(1 - y)}$$

$$\therefore C = \frac{y^4 + 11y^3 + 11y^2 + y}{(1 - 4y)(1 - y)^2}$$

$$Substitute\ C = \frac{y^4 + 11y^3 + 11y^2 + y}{(1 - 4y)(1 - y)^2}\ in\ (4.9), we\ have$$

$$B[1 - y] = \frac{y^4 + 11y^3 + 11y^2 + y}{(1 - 4y)(1 - y)^2}$$

$$\therefore B = \frac{y^4 + 11y^3 + 11y^2 + y}{(1 - 4y)(1 - y)^3}$$

$$Substitute\ B = \frac{y^4 + 11y^3 + 11y^2 + y}{(1 - 4y)(1 - y)^3}\ in\ (4.8), we\ have$$

$$A[1 - y] = \frac{y^4 + 11y^3 + 11y^2 + y}{(1 - 4y)(1 - y)^3}$$

$$\therefore A = \frac{y^4 + 11y^3 + 11y^2 + y}{(1-4y)(1-y)^4}$$

$$Substitute\ A = \frac{y^4 + 11y^3 + 11y^2 + y}{(1-4y)(1-y)^4}\ in\ (4.7), we\ have$$

$$g_4(y)[1-y] = 4\left(\frac{y^4 + 11y^3 + 11y^2 + y}{(1-4y)(1-y)^4}\right) = \frac{4(y^4 + 11y^3 + 11y^2 + y)}{(1-4y)(1-y)^4}$$

$$g_4(y)[1-y] = \frac{4y(y^3 + 11y^2 + 11y + 1)}{(1-4y)(1-y)^4}$$

$$\therefore g_4(y) = \frac{4y(y^3 + 11y^2 + 11y + 1)}{(1-4y)(1-y)^5}$$

Therefore, the closed form of the generating function for

$$f_{m+1} = \ 4(f_m + m^4)\ for\ all\ m \geq 0\ and\ f_0 = 0\ is$$

$$g_4(y) = \frac{4y(y^3 + 11y^2 + 11y + 1)}{(1-4y)(1-y)^5} = \frac{4yc_4(y)}{(1-4y)(1-y)^5}$$

$$\vdots$$

$f_{m+1} = k(f_m + m^k)\ for\ all\ m \geq 0, k > 0\ and\ f_0 = 0$

The sequence of

$$f_{m+1} = k(f_m + m^k) = \{k \cdot 0^k, k \cdot 1^k, (k^2 \cdot 1^k + k \cdot 2^k), (k^3 \cdot 1^k + k^2 \cdot 2^k + k \cdot 3^k), (k^4 \cdot 1^k + k^3 \cdot 2^k + k^2 \cdot 3^k + k \cdot 4^k) + \cdots\}$$

$$f_{m+1} = \sum_{j=1}^{m-1} k^{m-j} j^k$$

Applying the technique of differencing in generating function to get its closed form, we have

$$g_k(y) = \ \frac{ky[c_k(y)]}{(1-ky)(1-y)^{k+1}}$$

$$\therefore The\ closed\ form\ of\ the\ generating\ functions\ for$$

$$f_{m+1} = k(f_m + m^k)\ for\ all\ m \geq 0, k > 0\ and\ f_0 = 0\ is$$

$$g_k(y) = \ \frac{ky[c_k(y)]}{(1-ky)(1-y)^{k+1}}$$

where $(k-1)$ degree polynomial is $c_k(y)$. Now, let us discuss the nature of this polynomials as shown below.

4.1.1. Discussion of the result

The polynomials of the numerators from number 4.1 (1) to number 4.1 (4)) since the denominator has a clear pattern, can be explained using the pyramid below

$$y(1)$$

$$2y(y+1)$$

$$3y(y^2+4y+1)$$

$$4y(y^3+11y^2+11y+1)$$

The polynomials inside the brackets resemble Euler's polynomials which were obtained from Euler's number as shown in table 3.1 above. The only difference is, we have y in front of the brackets in the first line, $2y$ in front of the brackets in the second line, $3y$ in front of the brackets in the third line, $4y$ in front of the brackets in the fourth line and so on. Thus, we shall now apply the formula of Euler's number to generalize the technique for constructing the closed form of the generating functions of sequences of the form $f_{m+1} = k(f_m + m^k)\ for\ all\ m \geq 0, k > 0\ and\ f_0 = 0$ since these polynomials resemble Eulerian polynomials.

Therefore, the generalized technique for constructing the closed form of the generating functions of sequences of the form

$$f_{m+1} = k(f_m + m^k)\ for\ all\ m \geq 0, k > 0\ and\ f_0 = 0$$

is

$$g_k(y) = \frac{ky[c_k(y)]}{(1-ky)(1-y)^{k+1}} = \frac{ky\sum_{n=0}^{k-1} C(k,n)y^n}{(1-ky)(1-y)^{k+1}}$$

where $C(k,n)$ can be gotten from Euler's numbers and its formula is

$$C(k,n) = \sum_{i=0}^{n}(-1)^i\binom{k+1}{i}(n+1-i)^k\ as\ we\ illustrated\ in\ (3.1).$$

Remark: We can comfortably test the efficiency of this generalized technique we developed for obtaining the closed form of the generating function for the sequence of the form $f_{m+1} = k(f_m + m^k)\ for\ all\ m \geq 0, k > 0\ and\ f_0 = 0$, by using it to solve some mathematical problem.

Example 4.1 Obtain the closed form of the generating function for the sequence $f_{m+1} = 4(f_m + m^4)\ for\ all\ m \geq 0\ and\ f_0 = 0$

Solution

To obtain the closed form of the generating function for the sequence $f_{m+1} = 4(f_m + m^4)\ for\ all\ m \geq 0\ and\ f_0 = 0$, we shall use the generalized technique we developed above. Thus, the generalized technique is

$$g_k(y) = \frac{ky\sum_{n=0}^{k-1} C(k,n)y^n}{(1-ky)(1-y)^{k+1}}$$

Here, $k = 4$. Thus, we consider only the numerator since the denominator has a clear pattern. We have

$$4y\sum_{n=0}^{4-1} C(4,n)y^n = 4y\sum_{n=0}^{3} C(4,n)y^n = 4y[C(4,0)y^0 + C(4,1)y^1 + C(4,2)y^2 + C(4,3)y^3]$$

$$where\ C(4,0) = \sum_{i=0}^{0}(-1)^i\binom{4+1}{i}(0+1-i)^4 = (-1)^0\binom{5}{0}(0+1-0)^4 = 1(1)(1) = 1$$

$$C(4,1) = \sum_{i=0}^{1}(-1)^i\binom{4+1}{i}(1+1-i)^4$$

$$= (-1)^0 \binom{5}{0} (1+1-0)^4 + (-1)^1 \binom{5}{1} (1+1-1)^4$$

$$= 1(1)(2)^4 + (-1)(5)(1)^4 = 16 + (-5) = 16 - 5 = 11$$

$$\therefore C(4,1) = 11$$

$$C(4,2) = \sum_{i=0}^{2} (-1)^i \binom{4+1}{i} (2+1-i)^4$$

$$= (-1)^0 \binom{5}{0} (2+1-0)^4 + (-1)^1 \binom{5}{1} (2+1-1)^4 + (-1)^2 \binom{5}{2} (2+1-2)^4$$

$$= (1)(1)(3)^4 + (-1)(5)(2)^4 + (1)(10)(1)^4$$

$$= 81 - 80 + 10 = 11$$

$$\therefore C(4,2) = 11$$

$$C(4,3) = \sum_{i=0}^{3} (-1)^i \binom{4+1}{i} (3+1-i)^4$$

$$= (-1)^0 \binom{5}{0} (3+1-0)^4 + (-1)^1 \binom{5}{1} (3+1-1)^4 + (-1)^2 \binom{5}{2} (3+1-2)^4 + (-1)^3 \binom{5}{3} (3+1-3)^4$$

$$= (1)(1)(4)^4 + (-1)(5)(3)^4 + (1)(10)(2)^4 + (-1)(10)(1)^4$$

$$= 256 - 405 + 160 - 10 = 416 - 415 = 1$$

$$\therefore C(4,3) = 1$$

$$\therefore 4y \sum_{n=0}^{4-1} C(4,n) y^n = 4y \sum_{n=0}^{3} C(4,n) y^n = 4y[C(4,0)y^0 + C(4,1)y^1 + C(4,2)y^2 + C(4,3)y^3]$$

$$= 4y[1y^0 + 11y^1 + 11y^2 + 1y^3]$$

$$= 4y[1 + 11y + 11y^2 + y^3]$$

Thus, the closed form of the generating function for the sequence

$f_{m+1} = 4(f_m + m^4)\ for\ all\ m \geq 0\ and\ f_0 = 0$ is

$$g_4(y) = \frac{ky\sum_{n=0}^{k-1} C(k,n)y^n}{(1-ky)(1-y)^{k+1}} = \frac{4y(1+11y+11y^2+y^3)}{(1-4y)(1-y)^{4+1}} = \frac{4y(1+11y+11y^2+y^3)}{(1-4y)(1-y)^5}$$

Hence, the closed form of the generating function for the sequence

$$f_{m+1} = 4(f_m + m^4)\ is$$

$$g_4(y) = \frac{4y(1+11y+11y^2+y^3)}{(1-4y)(1-y)^5} = \frac{4y(y^3+11y^2+11y+1)}{(1-4y)(1-y)^5}$$

This result is the same with the result we have in number 4.1.1 (4).

4.2 Similarly, the generalized technique for constructing the closed form of the generating functions of sequences of the form

$$\boldsymbol{f_{m+1} = k\big(f_m + (m-1)^k\big) for\ all\ m \geq 1, k > 0\ and\ f_1 = 0\ is}$$

$$\boldsymbol{g_k(y) = \frac{ky^2 \sum_{n=0}^{k-1} C(k,n) y^n}{(1-ky)(1-y)^{k+1}}}$$

where $\boldsymbol{C(k,n)}$ is Euler's numbers and its formula is

$$\boldsymbol{C(k,n) = \sum_{i=0}^{n} (-1)^i \binom{k+1}{i} (n+1-i)^k\ as\ we\ illustrated\ in\ (3.2).}$$

Example 4.2 Obtain the closed form of the generating function for the sequence $f_{m+1} = 4(f_m + (m-1)^4)\ for\ all\ m \geq 1\ and\ f_1 = 0$

Solution

Now, we can use the above generalized technique to obtain the closed form of the generating function of $f_{m+1} = 4(f_m + (m-1)^4)$. Here, we observe that $k = 4$. Thus, we obtain the closed form of the generating function for the sequence $f_{m+1} = 4(f_m + (m-1)^4)\ for\ all\ m \geq 1\ and\ f_1 = 0$ using our generalized technique

$$g_k(y) = \frac{ky^2 \sum_{n=0}^{k-1} C(k,n) y^n}{(1-ky)(1-y)^{k+1}}$$

Now, we consider only the numerator since the denominator has a clear pattern. We have

$$4y^2 \sum_{n=0}^{4-1} C(4,n) y^n = 4y^2 \sum_{n=0}^{3} C(4,n) y^n = 4y^2[C(4,0)y^0 + C(4,1)y^1 + C(4,2)y^2 + C(4,3)y^3]$$

$$where\ C(4,0) = \sum_{i=0}^{0} (-1)^i \binom{4+1}{i} (0+1-i)^4 = (-1)^0 \binom{5}{0} (0+1-0)^4 = 1(1)(1) = 1$$

$$C(4,1) = \sum_{i=0}^{1} (-1)^i \binom{4+1}{i} (1+1-i)^4$$

$$= (-1)^0 \binom{5}{0} (1+1-0)^4 + (-1)^1 \binom{5}{1} (1+1-1)^4$$

$$= 1(1)(2)^4 + (-1)(5)(1)^4 = 16 + (-5) = 16 - 5 = 11$$

$$\therefore C(4,1) = 11$$

$$C(4,2) = \sum_{i=0}^{2} (-1)^i \binom{4+1}{i} (2+1-i)^4$$

$$= (-1)^0 \binom{5}{0} (2+1-0)^4 + (-1)^1 \binom{5}{1} (2+1-1)^4 + (-1)^2 \binom{5}{2} (2+1-2)^4$$

$$= (1)(1)(3)^4 + (-1)(5)(2)^4 + (1)(10)(1)^4$$

$$= 81 - 80 + 10 = 11$$

$$\therefore C(4,2) = 11$$

$$C(4,3) = \sum_{i=0}^{3} (-1)^i \binom{4+1}{i} (3+1-i)^4$$

$$= (-1)^0 \binom{5}{0} (3+1-0)^4 + (-1)^1 \binom{5}{1} (3+1-1)^4 + (-1)^2 \binom{5}{2} (3+1-2)^4 + (-1)^3 \binom{5}{3} (3+1-3)^4$$

$$= (1)(1)(4)^4 + (-1)(5)(3)^4 + (1)(10)(2)^4 + (-1)(10)(1)^4$$

$$= 256 - 405 + 160 - 10 = 416 - 415 = 1$$

$$\therefore C(4,3) = 1$$

$$\therefore 4y^2 \sum_{n=0}^{4-1} C(4,n)y^n = 4y^2 \sum_{n=0}^{3} C(4,n)y^n = 4y^2[C(4,0)y^0 + C(4,1)y^1 + C(4,2)y^2 + C(4,3)y^3]$$

$$= 4y^2[1y^0 + 11y^1 + 11y^2 + 1y^3]$$

$$= 4y^2[1 + 11y + 11y^2 + y^3]$$

Thus, the closed form of the generating function for the sequence

$f_{m+1} = 4(f_m + (m-1)^4)\ for\ all\ m \geq 1\ and\ f_1 = 0$

is

$$g_4(y) = \frac{ky^2 \sum_{n=0}^{k-1} C(k,n)y^n}{(1-ky)(1-y)^{k+1}} = \frac{4y^2(1 + 11y + 11y^2 + y^3)}{(1-4y)(1-y)^{4+1}} = \frac{4y^2(1 + 11y + 11y^2 + y^3)}{(1-4y)(1-y)^5}$$

Hence, the closed form of the generating function for the sequence

$$f_{m+1} = 4(f_m + (m-1)^4)\ for\ all\ m \geq 1\ and\ f_1 = 0$$

is

$$g_4(y) = \frac{4y^2(1 + 11y + 11y^2 + y^3)}{(1-4y)(1-y)^5}$$

Note: The difference between the closed form of the generating functions of sequences of the form $f_{m+1} = k(f_m + m^k)\ for\ all\ m \geq 0, k > 0\ and\ f_0 = 0$" and closed form of the generating functions of sequences of the form

$$\boldsymbol{f_{m+1} = k\left(f_m + (m-1)^k\right)\ for\ all\ m \geq 1, k > 0\ and\ f_1 = 0\text{" } is;}$$

The closed form of the generating functions of sequences of the form

$$f_{m+1} = k(f_m + m^k)\ for\ all\ m \geq 0, k > 0\ and\ f_0 = 0$$

is one step method while the closed form of the generating functions of sequences of the form

$$f_{m+1} = k(f_m + (m-1)^k) for\ all\ m \geq 1, k > 0\ and\ f_1 = 0$$

is two steps method.

4.3 Applications of the technique of generating functions in solving some important problems of recurrence relations

Example 4.3 Prove that the generating function

$$x = \sum_{n\geq 0} a_n y^n$$

based on Fibonacci sequence satisfies the ordinary differential equation

$$x'' + bx' + cx = o \text{ } if \text{ } and \text{ } only \text{ } if \text{ } b = -(n+2) \text{ } and \text{ } c = -(n+1)(n+2).$$

Proof

$$Let \text{ } x = \sum_{n\geq 0} a_n y^n$$

$$Then, x' = \sum_{n\geq 0} n a_n y^{n-1} = \sum_{n\geq 1} n a_n y^{n-1} = \sum_{n\geq 0} (n+1) a_{n+1} y^n$$

$$x'' = \sum_{n\geq 2} n(n-1) a_n y^{n-2} = \sum_{n\geq 0} (n+1)(n+2) a_{n+2} y^n$$

($\Longrightarrow$) $\boldsymbol{if}$:

$$if \text{ } b = -(n+2) and \text{ } c = -(n+1)(n+2), then$$

$$x'' + bx' + cx = x'' - (n+2)x' - (n+1)(n+2)x$$

$$= \sum_{n\geq 0} (n+1)(n+2) a_{n+2} y^n - \sum_{n\geq 0} (n+1)(n+2) a_{n+1} y^n - \sum_{n\geq 0} (n+1)(n+2) a_n y^n$$

$$\sum_{n\geq 0} \{(n+1)(n+2) a_{n+2} - (n+1)(n+2) a_{n+1} - (n+1)(n+2) a_n\} y^n$$

$$\sum_{n\geq 0} \{(n+1)(n+2)[a_{n+2} - (a_{n+1} + a_n)]\} y^n$$

$$\sum_{n\geq 0} (n+1)(n+2) \cdot 0 \cdot y^n \text{ } since \text{ } a_{n+2} = a_{n+1} + a_n = 0$$

So that

$$x = \sum_{n\geq 0} a_n y^n$$

Satisfies the ordinary differential equation $x'' + bx' + cx = o$

($\Longleftarrow$) $\boldsymbol{only \text{ } if}$:

$$If \text{ } x = \sum_{n\geq 0} a_n y^n$$

satisfies the ordinary differential equation $x'' + bx' + cx = o$

$$then\ x'' + bx' + cx = o \Longrightarrow \sum_{n\geq 0}[(n+1)(n+2)a_{n+2} + b(n+1)a_{n+1} + ca_n]\, y^n = 0$$

So that

$$(n+1)(n+2)a_{n+2} + b(n+1)a_{n+1} + ca_n = 0.$$

But

$$a_{n+2} = a_{n+1} + a_n$$

so,

$$(n+1)(n+2)(a_{n+1} + a_n) + b(n+1)a_{n+1} + ca_n = 0$$

$$[(n+1)(n+2) + b(n+1)]a_{n+1} + [(n+1)(n+2) + c]a_n = 0$$

$$So, (n+1)(n+2) + b(n+1) = 0\ and\ (n+1)(n+2) + c = 0$$

$$Thus, b = -(n+2)\ and\ c = -(n+1)(n+2)$$

Hence,

$$x = \sum_{n\geq 0} a_n y^n\ satisfies\ the\ ordinary\ differential\ equation\ x'' + bx' + cx = 0$$

Only if $b = -(n+2)\ and\ c = -(n+1)(n+2)$.

Example 4.4 Show that the generating function $G(y)$ of the (generating) function

$$g(y) = \sum_{n\geq 0} ny^n$$

is the original (generating) function

$$g(y) = \sum_{n\geq 0} ny^n$$

Proof

$$g(y) = \sum_{n\geq 0} ny^n = \sum_{n\geq 1} ny^n = \sum_{n\geq 0}(n+1)y^{n+1}$$

$$g'(y) = \sum_{n\geq 0} n^2 y^{n-1} = \sum_{n\geq 0}(n+1)^2 y^n$$

$$g''(y) = \sum_{n\geq 0}(n+1)^2 n y^{n-1} = \sum_{n\geq 1} n(n+1)^2 y^{n-1} = \sum_{n\geq 0}(n+1)(n+2)^2 y^n$$

$$g'''(y) = \sum_{n\geq 0} n(n+1)(n+2)^2 y^{n-1} = \sum_{n\geq 0}(n+1)(n+2)(n+3)^2 y^n$$

$$g^{iv}(y) = \sum_{n\geq 0}(n+1)(n+2)(n+3)(n+4)^2 y^n$$

$$\vdots$$

$$g^{(r)}(y) = \sum_{n\geq 0}\prod_{j=1}^{r-1}(n+j)(n+r)^2 y^n$$

$$g^{(r)}(0) = r \cdot r!$$

$G(y) = \sum_{n\geq 0} \frac{g^{(r)}(0)}{n!} y^n = \sum_{n\geq 0} \frac{n \cdot n!}{n!} y^n = \sum_{n\geq 0} n y^n = g(y)$

5. Conclusion

Generating function technique provides a way of using a single function to encode an infinite sequence of numbers by treating them as the coefficients of a formal power series. It has proved to be very useful tools which facilitate the solution of various classes of counting problems as questions about the convergence are not required. The major idea of a generating function is to use a single function instead of an infinite sequence of numbers. In this study, the generalized techniques for constructing the closed form of the generating functions of the sequences of the form $f_{m+1} = k(f_m + m^k)\ for\ all\ m \geq 0, k > 0\ and\ f_0 = 0\ and\ \ f_{m+1} = k(f_m + (m-1)^k) for\ all\ m \geq 1, k > 0\ and\ f_1 = 0$ were developed. Lastly, the techniques of generating functions were applied to solve some important problems of recurrence relations.

Recommendation/Suggestion

Further research should be done on generating function of zeta and polylogarithmic function. Also, it would be of interest to obtain the closed form of the generating function for a sequence of the form $f_{m+1} = a(bf_m + cm^k)\ for\ all\ m \geq 0; k > 0; f_0 = 0;\ a, b, c$ constants which is more general than the one studied in this research.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to be disclosed.

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