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Development of a new exponential generalized family of distribution with its properties and application

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Abstract

In the field of reliability analysis, the selection of an appropriate lifespan model is critical. With a multitude of lifetime distributions accessible, the hunt for a more suited distribution remains essential. In this paper, we offer a unique class of distributions derived from the notion of exponential generalization, improved with changes to boost flexibility. Our suggested distribution incorporates multiple hazard rate profiles, giving enhanced flexibility. Analytical characteristics including the r^{th} moment, moment generating function, quantile function, distribution of order statistics, and Rényi entropy are obtained. Employing maximum likelihood estimation, we estimate the unknown parameters. Through simulation tests and analysis of real-world datasets, we exhibit the model's usefulness compared to five existing lifespan distributions, emphasising the better performance of the GAYUF distribution. This research underlines the GAYUF distribution as a better model in the field of lifespan analysis

Keywords: Probability Distribution; Hazard Rate; Statistical Properties; Exponential Generalization; Maximum Likelihood Estimation

1. Introduction

There are several typical theoretical distributions documented in the literature, including the Geometric distribution, Exponential distribution, Uniform distribution, Normal distribution, Gamma distribution, Beta distribution, and others. These standard distributions are well recognised for their significant relevance and extensive use in several fields of study. Various lifespan distributions, including exponential, Weibull, Gompertz, and Gumbel distributions, have been employed for simulating dependability, human mortality, and actuarial data. The Exponential distribution is commonly used to analyse lifespan data in numerous fields of research because to its simplicity and analytical tractability (Maurya, Kaushik, Singh & Singh, 2016).

Nevertheless, the exponential distribution exhibits the same characteristics of a constant failure rate and memorylessness as the geometric distribution (Oguntunde, 2017), which is the focus of this work. The existing literature has demonstrated that the one parameter exponential distribution possesses various intriguing characteristics. However, the failure rate property of the Exponential distribution renders it inappropriate for modelling real-life scenarios with bathtub (uni-antimodal) and inverted bathtub (unimodal) failure rates (Lemonte, 2013).

There are several techniques to propose a new distribution using some baseline distribution as found in the literature some of which include:

Gupta *et al*., (1998) have proposed the cumulative distribution function (cdf) $G_1(x)$ of new distribution corresponding to the cdf, $F(x)$ of baseline distribution as,

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$$G_1(x)=[f(x)]^a$$

where, a>0 is the shape parameter.

In a related development, Shaw and Buckley (2009) developed the quadratic rank transmutation map (QRTM) in order to form flexible distribution families by adding a new parameter to an existing distribution.

If $G_2(x)$ is the cumulative function of transmuted distribution consistent to the baseline distribution having F(x), then,

$$G_2(x)=(1+\lambda)F(x)-\lambda[F(x)]^2$$

Where $|\lambda|\leq 1$

Several generalisations have been proposed based on QRTM, including the transmuted extreme value distribution (Aryal and Tsokos, 2011), transmuted inverse Weibull distribution (Khan et al., 2014), transmuted modified Weibull distribution (Khan and King, 2013), transmuted log-logistic distribution (Aryal, 2013), transmuted exponential distribution (Kumar et al., 2015), cubic transmuted family of distribution (Aslam Mohammad et al., 2018), and Generalised Exponential Cubic Transmuted (Oseghale, 2021), among others.

Cordeiro et al., (2013) introduced a novel category of distribution by including two more shape factors. It is worth mentioning that almost all additions include just a small number of extra parameters to the existing model. The outcome gives rise to complications in future deductions. Put simply, including extra factors allows for more versatility, but it also increases the difficulty in estimating the parameter(s) (Maurya, et al., 2016). Considering this aspect, Kumar et al. (2015) introduced a transformation known as the DUS transformation to achieve a novel distribution. The DUS transformation yields a new cumulative distribution function (CDF) F(x) from the baseline CDF G(x) as shown below:

$$F(x)=\frac{\ell^{G(x)}-1}{\ell-1}$$

A number of DUS transformations have been developed, utilising exponential and Weibull distributions as the baseline distribution (Chaudhry & Shareef, 2021). Empirical evidence demonstrates that the newly generated data exhibits a distribution with hazard rates that vary over time. Another benefit of employing this transformation is that the resulting distribution maintains the trait of being parsimonious in parameterization, as it does not introduce any more parameters.

In a correlated advancement, (Maurya, et al., 2016) suggested the utilisation of DUS transformation on the exponentiated cumulative distribution function (cdf), subsequently denoted as Generalised DUS (GDUS) transformation. Consider X as a random variable with cumulative distribution function (cdf) G(x), and let g(x) represent the appropriate probability density function (pdf) considered as the baseline distribution. If F(x) and f(x) represent the cumulative distribution function (CDF) and probability density function (PDF) of the proposed GDUS distribution, respectively, then F(x) is defined as:

$$F(x)=\frac{\ell(G^{\alpha}(x))-1}{\ell-1}$$

The suggested distribution can handle hazard rates of many forms, including the bathtub shape.

The exponential distribution has been selected as the reference distribution.

The purpose of this paper is to introduce a novel distribution class that encompasses all possible hazard rates, based on the selection of the shape parameter and size of the parameters. We suggest employing the AYUF transformation on the exponential cumulative distribution function (CDF), which will be referred to as the Generalised AYUF (GAYUF) transformation. The resulting distribution is anticipated to exhibit both monotonic and bathtub-shaped hazard rates, contingent upon the selection of the values of the parameters. The new distribution using the (GAYUF) transformation is defined as follows:

2. The new Generalized AYUF (GAYUF) transformation

Let X be a random variable with cumulative distribution function $G(x)$ and $g(x)$ be the corresponding probability distribution function taken as the baseline distribution. And if $F(x)$ and $f(x)$ be the CDF and PDF of the proposed distribution. The proposed AYUF generated family of distribution is given by; $F(x,\lambda,\alpha,\theta)=1-\ell^{-(x\lambda+\alpha G(x))^{\theta}}\ for x\geq 0,\lambda>0,\alpha>0 and\theta>0$ (1)

and the corresponding pdf is,

$$f(x,\lambda,\alpha,\theta)=\theta(\lambda+\alpha g(x))(\lambda x+\alpha G(x))^{\theta-1}\left[\ell^{-(x\lambda+\alpha G(x))^{\theta}}\right] \quad (2)$$

Where θ is the shape parameter.

2.1. GAYUF CDF Validation

However, to show that the CDF in equation is valid (1), we need to show that it satisfies the necessary properties of a valid CDF for all parameter values λ,α,θ and for all values of x. The necessary properties of a CDF are:

Non-Negativity: $F(x,\lambda,\alpha,\theta)\geq 0$ for all χ

Limits of infinity: $\underset{\chi\to-\infty}{Lim}F(x,\lambda,\alpha,\theta)=0$ and $\underset{\chi\to+\infty}{Lim}F(x,\lambda,\alpha,\theta)=1$

Monotonicity $F(x_1,\lambda,\alpha,\theta)\leq F(x_2,\lambda,\alpha,\theta)$ for all $\chi_1\leq\chi_2$

Right-continuity: $F(\chi,\lambda,\alpha,\theta)$ is a right continuous function.

2.1.1. Proof Condition (I)

Since $G(x)$ is a CDF and CDFs are always non-decreasing and bounded between 0 and 1, and because x,λ,α are real numbers, $\lambda(x)$ and $G(x)$ are both real numbers.

Hence, $\lambda(x)+\alpha G(x)$ is real for all x . Raising this expressing to the power θ makes it non-negative. So, the exponential term is always non negative, and $F(x,\lambda,\alpha,\theta)$ is non-negative.

2.1.2. Condition (II)

As x approaches negative infinity, both $\lambda(x)$ and $\alpha G(x)$ approach negative infinity, and their sum remains negative. Raising a negative number to any power $\theta>0$ results in 0 as the limit of the exponential term. Therefore, $\underset{\chi\to-\infty}{Lim}F(x,\lambda,\alpha,\theta)=1-0=1$

As χ approaches positive infinity, both $\lambda(x)$ and $\alpha G(x)$ approach positive infinity, and their sum remains positive. Raising a positive number to any power $\theta>0$ results in a positive value.

Therefore, $\underset{\chi\to+\infty}{Lim}F(x,\lambda,\alpha,\theta)=1-\ell^{0}=1-1=0$. Hence the limit as satisfied.

2.1.3. Condition (III)

To check for monotonicity we need to verify that $F(x_1,\lambda,\alpha,\theta)\leq F(x_2,\lambda,\alpha,\theta)$ for all $x_1\leq x_2$.

Comparing the values of $[\lambda(x_1)+\alpha G(x_1)]^\theta$ and $[\lambda(x_2)+\alpha G(x_2)]^\theta$ and their corresponding exponential terms. Since exponential function is monotonically increasing, if $[\lambda(x_1)+\alpha G(x_1)]^\theta \leq [\lambda(x_2)+\alpha G(x_2)]^\theta$ then $F(x_1,\lambda,\alpha,\theta) \leq F(x_2,\lambda,\alpha,\theta)$. Since the exponential function is monotonically increasing, $F(x_1,\lambda,\alpha,\theta) \leq F(x_2,\lambda,\alpha,\theta)$ so, the proposed CDF $F(x,\lambda,\alpha,\theta)$ is indeed monotonic as x, increases.

2.1.4. Condition (IV)

Right continuity is the property that ensures there are no jumps in the CDF. To check right continuity, we need to verify that $F(x,\lambda,\alpha,\theta)$ is continuous from the right at every point x . Continuity of $F(x,\lambda,\alpha,\theta)$ generally depends on the continuity of the function $\lambda(x)$ and $G(x)$. If $\lambda(x)$ and $G(x)$ are continuous functions of x , then the composition $\lambda(x)+\alpha G(x)$ is continuous, and $[\lambda(x)+\alpha G(x)]^\theta$ remains continuous. The exponential function is also continuous everywhere. Therefore, $F(x,\lambda,\alpha,\theta)$ is continuous from the right at every χ . Hence, the proposed cumulative distribution function $F(x,\lambda,\alpha,\theta)=1-\ell^{-(x\lambda+\alpha G(x))^\theta}$ $for x\geq 0, \lambda>0, \alpha>0 and \theta>0$ satisfies the properties of non-negativity limits at infinity, monotonicity and right continuity. These properties collectively established that it is a valid family of distribution indexed by parameter λ,α,θ. It conforms to the general criteria required for a cumulative distribution function, working it a valid distribution function for this family. Figures 1 to 4 provide the graphical representation of the PDF and failure rate function respectively for various parameter values for AYUF-Exponential and AYUF-Weibull.

However, analysis of the density graph for the two different scenarios reveals that the shape of GAYUF is quite versatile, exhibiting moderately positive skewness, approximate symmetry, and moderately negative skewness for different parameter values. This suggests that the distribution is a suitable model for the lifespan of components and systems, as well as other non-negative variables.

The cumulative distribution graph also indicates that GAYUF distribution has a proper pdf since it converges to one.

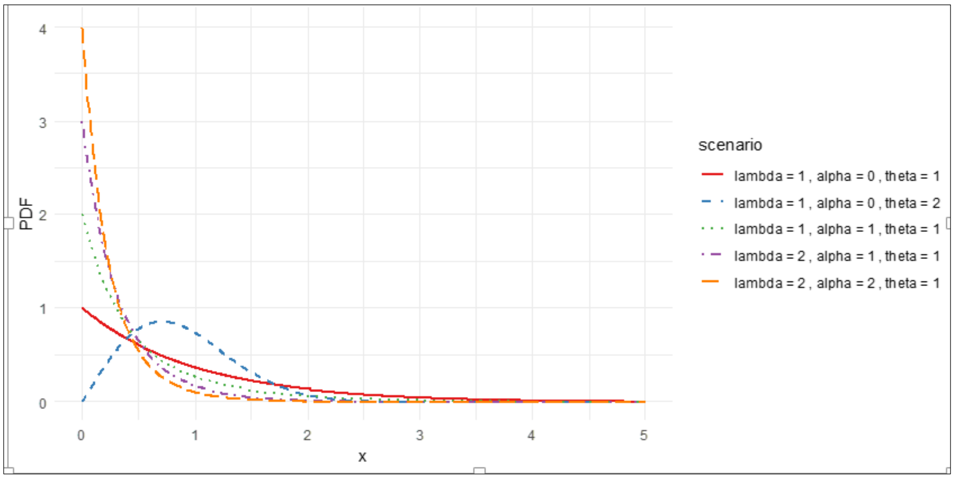

Figure 1 Density graph of AYUF-Exponential Distribution

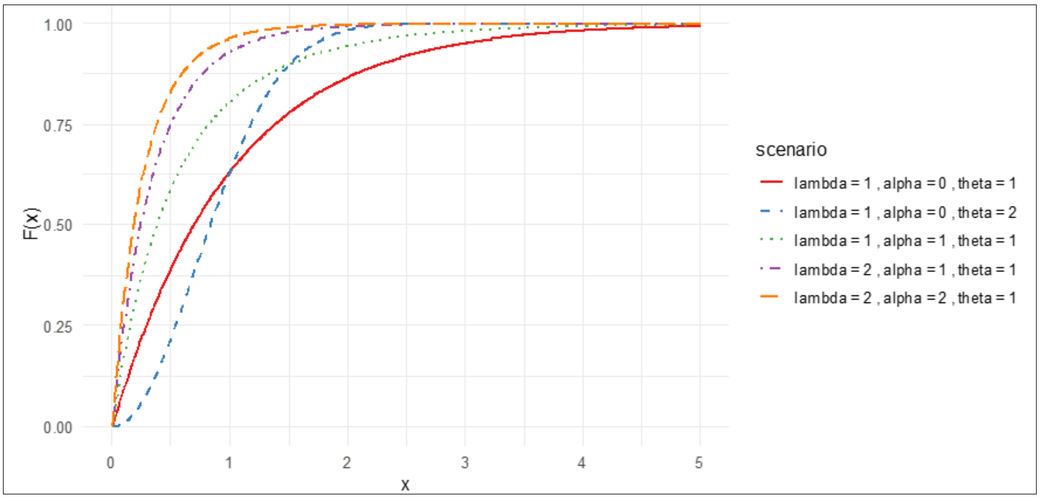

Figure 2 Cummulative Probability Graph of AYUF-Exponential

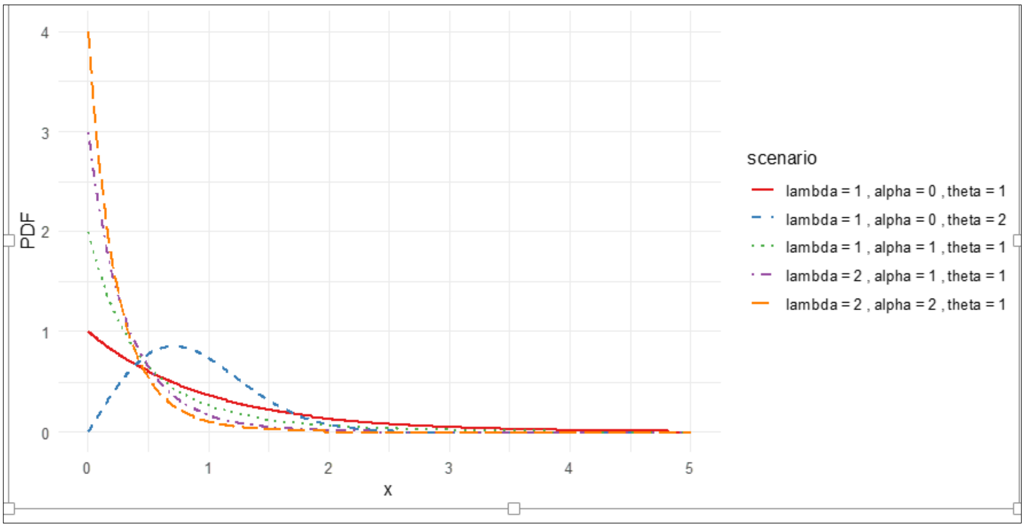

Figure 3 Density graph of AYUF Weibull Distribution

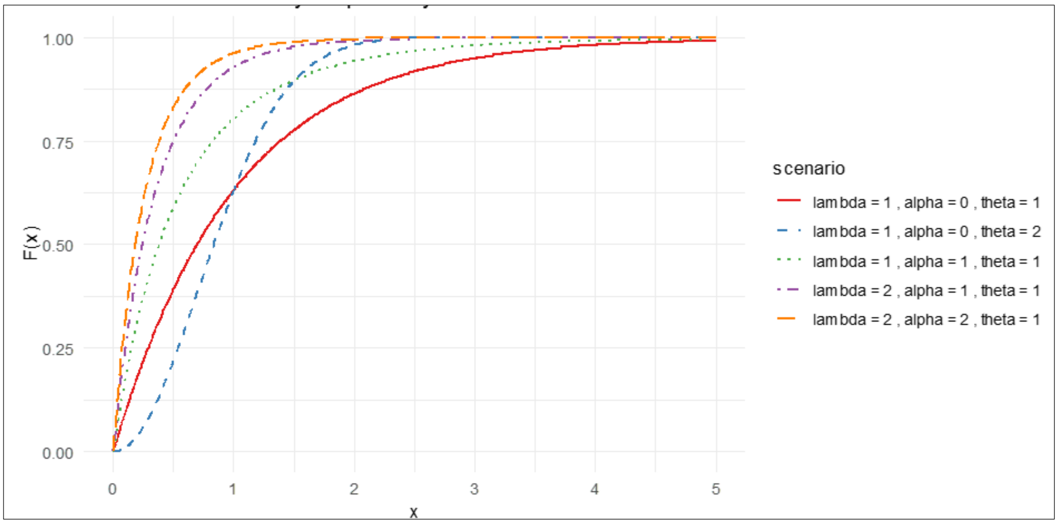

Figure 4 Cummulative Probability Graph of AYUF-Weibull Distribution

2.2. Survival Function

The survival function for the GAYUF denoted as $S(x)$

$$S(x) = 1 - F(\chi, \lambda, \alpha, \theta)$$

$$S(x) = \left\langle \ell^{-[\lambda x + \alpha G(x)]^{\theta}} \right\rangle \quad (3)$$

for $x \geq 0$, $\lambda > 0, \alpha > 0$ and $\theta > 0$

As indicated in figure 5 and 6, the survival probability distribution decreases as survival time increases.

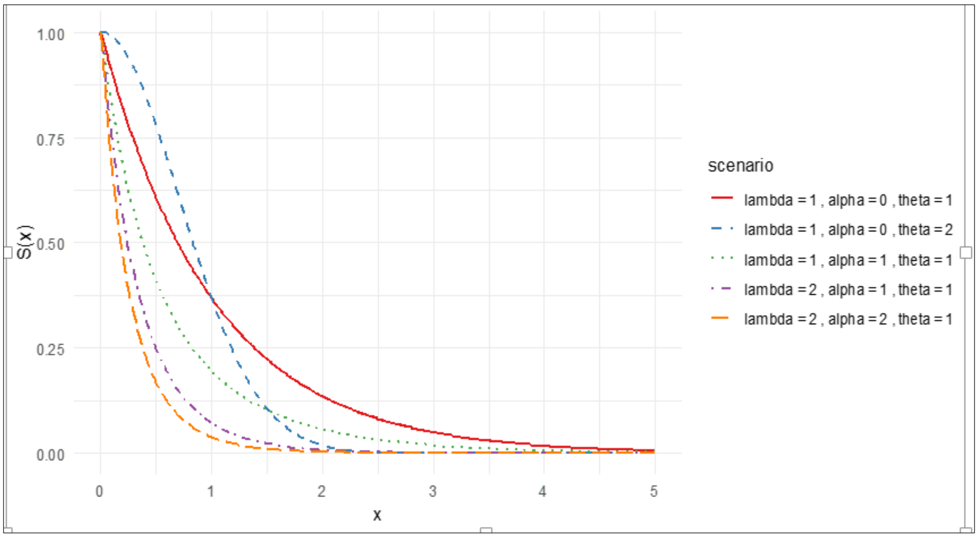

Figure 5 Survival Graph for AYUF-Exponential Distribution

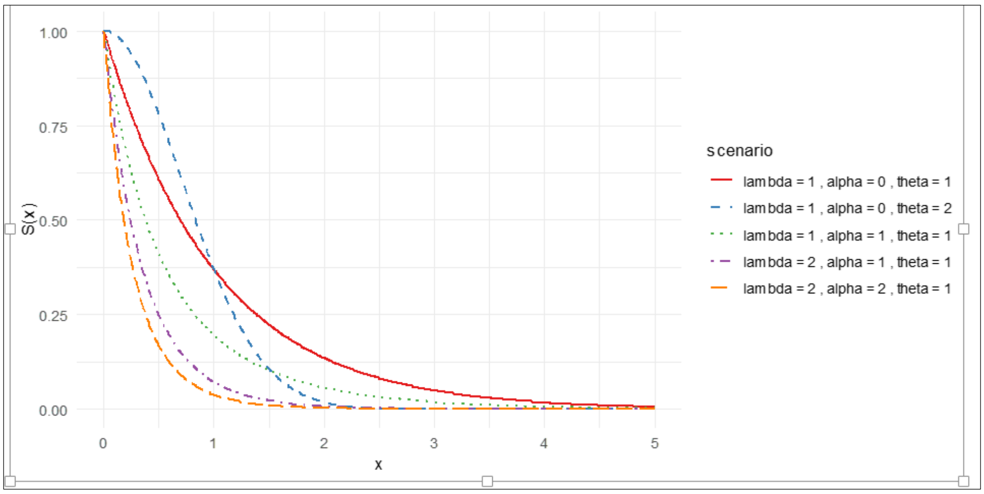

Figure 6 Survival Graph for AYUF-Weibull Distribution

2.3. Hazard Function

$$h(x)=\frac{f(x)}{S(x)}$$

$$h(x)=\frac{\theta(\lambda+\alpha g(\chi))(\lambda x+\alpha G(x))^{\theta-1}\left[\ell^{-(\chi\lambda+\alpha G(\chi))^{\theta}}\right]}{\left\langle \ell^{-[\lambda x+\alpha G(x)]^{\theta}}\right\rangle}$$

$$h(x)=\theta(\lambda x+\alpha G(x))^{\theta-1}(\lambda+\alpha g(\chi)) \quad (4)$$

2.3.1. Behaviour of the hazard rate of the proposed gayuf

- **Constant Hazard Rate** (θ):

When (θ) is 1, the hazard function becomes

$h(x)=(\lambda x+\alpha G(x))^{1-1}(\lambda+\alpha g(\chi))=\lambda+\alpha g(\chi)$. This indicates that the hazard rate is constant over time, as it is independent of (χ). This behavior is characteristic of a constant failure rate model, similar to the exponential distribution.

- **Decreasing Failure Rate** (λ,α):

If (λ,α) are both positive, the hazard function can be negative. This occurs when c dominates $(\lambda+\alpha g(\chi))$ resulting in a negative hazard rate. In reliability situations, this implies a decreasing failure rate over time, where early failures are more common and the likelihood of failure decreases as time progresses.

- **Increasing Failure Rate** (λ,α):

Conversely, when (λ,α), are both positive, the hazard function can be positive. This occurs when $(\lambda+\alpha g(\chi))$ dominates $(\lambda x+\alpha g(\chi))$, resulting in a positive hazard rate. This implies an increasing failure rate over time, observed in scenarios where wear and tear lead to a growing risk of failure as time passes.

- **Magnitude of** (θ):

The magnitude of θ affects the steepness of the hazard function. A larger θ results in a more significant impact on the hazard rate, influencing how quickly the hazard rate changes with respect to χ. A smaller θ leads to a shallower slope, indicating a slower change in the hazard rate over time.

Hence, the above family of distribution can exhibit different failure rate behabiours depending on the values of $\lambda,\alpha\ \ \theta$, and the behavior of the baseline distribution CDF $G(x)$ and PDF $g(\chi)$. Each of these behavior has unique implications for reliability and survival analysis. The figure 7 8 indicates that the GAYUF for the two distributions is an increasing, constant and reversible function. This shows that the distribution is a flexible distribution that can be used to model real life events that exhibits non-monotone failure rate.

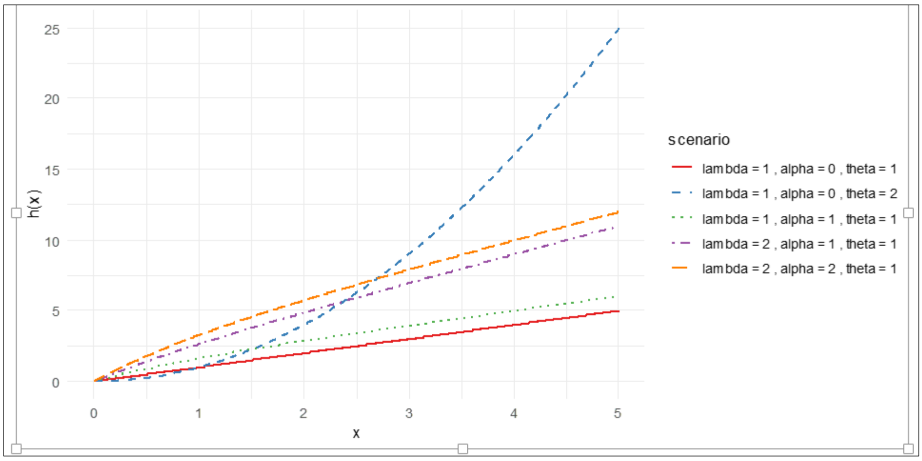

Figure 7 Hazard Graph for AYUF-Exponential Distribution

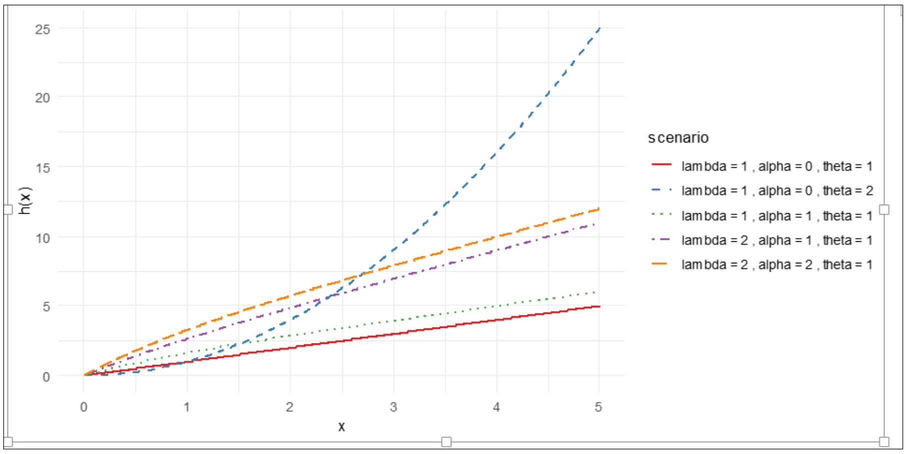

Figure 8 Hazard Graph for AYUF-Weibull Distribution

3. Deriving the Various Statistical Properties of the Proposed GAYUF

Several statistical properties of our proposed distribution like, r^{th} Moment, Moment Generating Function, Renyl Entropy, Quartile Function and Order Statistics are discussed below

3.1. r^{th} Moment

To derive the expression for the r^{th} moment of the distribution,

$$M_r = \int_{-\infty}^{\infty} x^r . f(x) \partial x$$

$$M_r = \int_{-\infty}^{\infty} x^r . \theta(\lambda + \alpha g(x))(\lambda x + \alpha G(x))^{\theta - 1}\left[\ell^{-(x\lambda + \alpha G(x))^{\theta}}\right]\partial x$$

Applying Lemma

$$= \int_0^{\infty} x^r . \left[\ell^{-\left(x\lambda^{\theta}\right)}\right]\partial x = \frac{1}{\theta}\lambda^{-\frac{r+1}{\theta}}\Gamma\left(\frac{r+1}{\theta}\right)$$

$$M_r = \int_0^\infty (\lambda x + \alpha G(x))^r . \theta(\lambda + \alpha g(x)) \ell^{-(x\lambda + \alpha G(x))^\theta} (\lambda x + \alpha G(x))^{\theta - 1} \partial x$$

$$M_r = \theta \int_0^\infty \ell^{-(x\lambda + \alpha G(x))^\theta} (\lambda x + \alpha G(x))^{r + \theta - 1} (\lambda + \alpha g(x)) \partial x$$

This integral is in the form of the lemma, with λ as the parameter λ, $(x\lambda + \alpha G(x))^\theta$ as the function x and $r + \theta - 1$ as the power.

$$M_r = \frac{1}{\theta} \lambda^{-\frac{r+\theta}{\theta}} \Gamma\left(\frac{r+\theta}{\theta}\right) \tag{5}$$

3.2. Moment Generating Function

$$M_X(t) = E[e^{tx}]$$

$$M_X(t) = \int_{-\infty}^\infty e^{t\chi} . f(x) \partial x$$

$$M_X(t) = \int_{-\infty}^\infty e^{t\chi} . \theta(\lambda + \alpha g(\chi))(\lambda x + \alpha G(x))^{\theta - 1} \ell^{-(\chi\lambda + \alpha G(\chi))^\theta} \partial \chi$$

$$= \theta \int_{-\infty}^\infty (\lambda + \alpha g(x))(\lambda x + \alpha G(x))^{\theta - 1} \ell^{-(t - x\lambda + \alpha G(x))^\theta} \partial x$$

$$= \theta \int_{-\infty}^\infty (\lambda + \alpha g(\chi))(\lambda x + \alpha G(x))^{\theta - 1} \ell^{t\chi + (-\chi\lambda - \alpha G(\chi))^\theta} \partial \chi$$

$$\int e^{t\chi} e^{-\lambda \chi^\theta} \partial x = \frac{1}{\theta} \lambda^{-\frac{1}{\theta}} \Gamma\left(\frac{1}{\theta}\right) \left(1 - \frac{t}{\lambda^{\frac{1}{\theta}}}\right)^{-\frac{1}{\theta}}$$

Notice that this is of the form $e^{-\lambda x^\theta}$, where $x = x\lambda + \alpha G(x)$

Substitute $x = x\lambda + \alpha G(x)$, then $dx = \frac{1}{\lambda} \partial x$

$$M_x(t) = \theta \int_0^\infty e^{t\left(\frac{x - \alpha G(x)}{\lambda}\right)} . \frac{\lambda}{\theta} (\lambda + \alpha g(x)) e^{-x^\theta} x^{\theta - 1} \partial x \quad = \frac{\theta}{\lambda} \int_0^\infty e^{\left(\frac{t}{\lambda} - 1\right)x} (\lambda + \alpha g(x)) e^{-x^\theta} x^{\theta - 1} \partial x$$

This integral is in the form of the lemma, with λ as the parameter λ, x as the function x and θ as the power.

$$M_X(t)=\frac{1}{\theta}\lambda^{-\frac{1}{\theta}}\Gamma\left(\frac{1}{\theta}\right)\left(1-\frac{\left(\frac{t}{\lambda}-1\right)}{\lambda^{\frac{1}{\theta}}}\right)^{-\frac{1}{\theta}} \qquad =\frac{1}{\theta}\lambda^{-\frac{1}{\theta}}\Gamma\left(\frac{1}{\theta}\right)\left(\frac{\lambda-t}{\lambda}\right)^{-\frac{1}{\theta}}$$

(6)

3.3. Renyl Entropy

The Renyl entropy of order is defined as:

$$H_\alpha(X)=\frac{1}{1-\alpha}\log\left(\int_{-\infty}^{\infty} f(x)^\alpha \partial x\right)$$

$$H_\alpha(X)=\frac{1}{1-\alpha}\log\left(\int_{-\infty}^{\infty}[\theta(\lambda+\alpha g(x))(x\lambda+\alpha G(x))]^{\alpha\theta-1}e^{-(x\lambda+\alpha G(x))^\theta}\partial x\right)$$

$$\int_{-\infty}^{\infty}[\theta(\lambda+\alpha g(x))(x\lambda+\alpha G(x))]^{\alpha\theta-1}e^{-(x\lambda+\alpha G(x))^\theta}\partial x = \int_{-\infty}^{\infty}[\theta(x\lambda+\alpha G(x))]^{\alpha\theta-1}e^{-(x\lambda+\alpha G(x))^\theta}\partial x$$

Now, let's define:

$$h(x)=\theta(x\lambda+\alpha G(x))= \int_{-\infty}^{\infty} h(x)^{\alpha\theta-1}e^{-h(x)^\theta}\partial x$$

We can rewrite the Renyl entropy using the gamma function lemma, which states that for $\alpha > 0$:

$$\int_0^\infty x^{\alpha-1}e^{-tx}\partial x=\frac{\Gamma(\alpha)}{t^\alpha}$$

Where $\Gamma(\alpha)$ is the gamma function,

Rewrite the integral using this lemma:

$$\int_{-\infty}^{\infty} h(x)^{\alpha\theta-1}e^{-h(x)^\theta}\partial x = \int_{-\infty}^{\infty} e^{(\alpha\theta-1)\log(h(x))-h(x)^\theta}\partial x = \int_{-\infty}^{\infty} e^{(\alpha\theta-1)\log(\theta(x\lambda+\alpha G(x)))-\theta(x\lambda+\alpha G(x))^\theta}\partial x$$

$$u'(x)=\frac{\theta'(x\lambda+\alpha G(x))}{\theta(x\lambda+\alpha G(x))}(\lambda+\alpha g(x)) \text{ and } v'(x)-\theta'(x\lambda+\alpha G(x))$$

$$=\int_{-\infty}^{\infty} e^{(\alpha\theta-1)\log(\theta(x\lambda+\alpha G(x)))-\theta(x\lambda+\alpha G(x))^\theta}\partial x = \int_{-\infty}^{\infty} e^{\alpha\theta u(x)-v(x)^\theta}\partial x$$

Using the lemma, we get:

$$=\frac{1}{\theta^{\alpha\theta}}\int_{-\infty}^{\infty}e^{\alpha u(x)-v(x)^{\theta}}\partial x=\frac{1}{\theta^{\alpha\theta}}\int_{-\infty}^{\infty}e^{(\alpha\theta-1)\log(\theta(x\lambda+\alpha G(x)))-\theta(x\lambda+\alpha G(x))^{\theta}}\partial x=\frac{1}{\theta^{\alpha\theta}}\frac{\Gamma(\alpha\theta)}{(\theta)^{\alpha\theta}}=\frac{\Gamma(\alpha\theta)}{\theta^{\alpha\theta}}$$

Substituting back into the Renyl entropy formula, we get:

$$H_{\alpha}(X)=\frac{1}{1-\alpha}\log\left(\frac{\Gamma(\alpha\theta)}{\theta^{\alpha\theta}}\right)=\frac{1}{1-\alpha}\log(\Gamma(\alpha\theta))-\log(\theta^{\alpha\theta})==\frac{1}{1-\alpha}\log(\Gamma(\alpha\theta))-\alpha\theta\log(\theta) \tag{7}$$

3.4. Quartile Function

$$F(\chi)=\int_{-\infty}^{x}f(x)\partial x$$

Given

$$f(x)=\theta(\lambda+\alpha g(x))(x\lambda+\alpha G(x))^{\theta-1}e^{-(x\lambda+\alpha G(x))^{\theta}}$$

$$F(x)=\int_{-\infty}^{x}\theta(\lambda+\alpha g(x))(x\lambda+\alpha G(x))^{\theta-1}e^{-(x\lambda+\alpha G(x))^{\theta}}\partial x$$

$$u=x\lambda+\alpha G(x)$$

$$\partial u=(x\lambda+\alpha g(x))\partial\chi$$

$$\partial\chi=\frac{1}{x\lambda+\alpha g(x)}\partial u$$

The integral becomes:

$$F(x)=\int_{-\infty}^{\chi}\theta u^{\theta-1}e^{-u\theta}\frac{1}{x\lambda+\alpha g(x)}\partial u$$

$$F(x)=\frac{1}{\lambda}\int_{-\infty}^{\chi}\theta u^{\theta-1}e^{-u\theta}\frac{1}{1+\frac{\alpha}{\lambda}g(x)}\partial u$$

If $G(x)$ is the cumulative distribution function (CDF) of some baseline distribution, denoted as $F_0(x)$ then $G(x)$ can be expressed as:

$$G(x)=F_0(x)$$

$$F(x)=\int_{-\infty}^{x}f(x)\partial x=\int_{-\infty}^{x}\theta(\lambda+\alpha f_0(x))(x\lambda+\alpha f_0(x))^{\theta-1}e^{-(x\lambda+\alpha f_0(x))^{\theta}}\partial x$$

$$u=x\lambda+\alpha F_0(x)$$

$$\partial u = \left(x\lambda + \alpha f_0(x)\right)\partial \chi$$

$$\partial x = \frac{1}{x\lambda + \alpha f_0(x)}\partial u$$

$$F(x) = \int_{-\infty}^{\chi} \theta u^{\theta-1} e^{-u\theta} \frac{1}{x\lambda + \alpha f_0(x)}\partial u = \frac{1}{\lambda}\int_{-\infty}^{\chi} \theta u^{\theta-1} e^{-u\theta} \frac{1}{1+\frac{\alpha}{\lambda} f_0(x)}\partial u = \int_{-\infty}^{\chi} \frac{\theta}{\lambda} \frac{u^{\theta-1}e^{-u\theta}}{1+\frac{\alpha}{\lambda} f_0(x)}\partial u$$

We can see that the integrand has the form $x^{\alpha-1}e^{-tx}$, which fits the lemma form.

$$x = u\theta$$

$$t = \frac{1}{\theta}$$

$$\partial x = \theta \partial u$$

Substituting these into the integral, we get:

$$F(x) = \int_{-\infty}^{\chi\theta} \frac{1}{\lambda} \frac{x^{\theta/\theta-1}e^{-x}}{1+\frac{\alpha}{\lambda} f_0(x)} \frac{1}{\theta}\partial x = \frac{1}{\theta\lambda}\int_{0}^{\chi\theta} \frac{x^{\theta-1}e^{-x}}{1+\frac{\alpha}{\lambda} f_0(x)} \frac{1}{\theta}\partial x$$

Use the lemma:

$$\int_0^{\infty} x^{\alpha-1}e^{-tx}\partial x = \frac{\Gamma(\alpha)}{t^{\alpha}}$$

Where $t = x\theta$ and $\alpha = \theta$ so, we have:

$$F(x) = \frac{1}{\theta\lambda}\frac{\Gamma(\theta)}{\left(x^{\theta}\right)^{\theta}} = \frac{1}{\lambda}\frac{\Gamma(\theta)}{\theta^{\theta}x^{\theta}}$$

Since $\lambda, \theta,$ and λ are all greater than 0, we can proceed with finding the quartile function $Q(p)$ which represents the value below which a certain proportion of the data falls.

From the cumulative distribution function (CDF) $F(x)$, we can find $Q(p)$ by solving the equation:

$$Q(p) = F^{-1}(p)$$

Where p is the desired quartile, and F^{-1} is the inverse of the CDF $F(x)$

Given that $F(x)$ is:

$$F(x) = \frac{1}{\lambda}\frac{\Gamma(\theta)}{\theta^{\theta}x^{\theta}}$$

We can solve x for in terms of p to find $Q(p)$:

$$p = \frac{1}{\lambda}\frac{\Gamma(\theta)}{\theta^{\theta}x^{\theta}} = \quad x^{\theta} = \frac{\lambda}{p}\frac{\Gamma(\theta)}{\theta^{\theta}} = \quad x = \left(\frac{\lambda}{p}\frac{\Gamma(\theta)}{\theta^{\theta}}\right)^{\frac{1}{\theta}}$$

To solve for $Q(p)$ we use the expression we derived earlier:

$$x = \left(\frac{\lambda}{p}\frac{\Gamma(\theta)}{\theta^{\theta}}\right)^{\frac{1}{\theta}}$$

Where p is the desired proportion, λ and θ are parameters, and $\Gamma(\theta)$ is the gamma function

$$Q(p) = \left(\frac{\lambda}{p}\frac{\Gamma(\theta)}{\theta^{\theta}}\right)^{\frac{1}{\theta}}$$

The quartile function $Q(p)$ in terms of $p, \lambda,$ and θ is:

$$Q(p) = \sqrt{\frac{\lambda}{4p}} = \left(\frac{\lambda}{p}\frac{\Gamma(\theta)}{\theta^{\theta}}\right)^{\frac{1}{\theta}}$$

Given $\lambda > 0, \theta > 0,$ and p as the desired proportion, we can solve this equation,

$$Q(p) = \left(\frac{\lambda}{p}\frac{\Gamma(\theta)}{\theta^{\theta}}\right)^{\frac{1}{\theta}}$$

$$Q(p) = \left(\frac{\lambda}{p}\frac{\Gamma(2)}{2^{2}}\right)^{\frac{1}{2}} = \left(\frac{\lambda}{p}\frac{1}{2}\right)^{\frac{1}{2}}$$

$$Q(p) = \left(\frac{\lambda}{4p}\right)^{\frac{1}{2}} \tag{8}$$

3.5. Order Statistics

Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function $f(\chi)$

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics, arranged in ascending order.

Let $F(x)$ be the cumulative distribution function (CDF) corresponding to $f(x)$

Let $f_{(k)}(X)$ be the density function of $X_{(k)}$

The order statistics are defined as follows:

$X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$

Lemma: Let X have probability density function $f(x)$ and cumulative distribution function $F(x)$

Then the density function of the order statistic $X_{(k)}$ is given by:

$$f_{(k)}(x) = \frac{n!}{(k-1)!(n-k)!}[F(x)]^{k-1}[1-F(x)]^{n-k} f(x)$$

Given

$$f(x) = \theta(\lambda + \alpha g(x))(x\lambda + \alpha G(x))^{\theta-1} e^{-(x\lambda + \alpha G(x))^{\theta}}$$

$$F(x) = \int_{-\infty}^{x} f(t)\partial t$$

Let's denote:

$$u = x\lambda + \alpha G(x)$$

$$\partial u = (x\lambda + \alpha g(x))\partial \chi$$

$$F(x) = \int_{-\infty}^{\chi} \theta u^{\theta-1} e^{-u\theta} \partial u$$

$$= 1 - e^{-\theta(\chi\lambda + \alpha G(\chi))}$$

$$f(x) = \theta(\lambda + \alpha g(x))(x\lambda + \alpha G(x))^{\theta-1} e^{-(x\lambda + \alpha G(x))^{\theta}}$$

Let's differentiate $F(x)$ and $f(x)$

$$\frac{\partial F}{\partial x} = \theta e^{-\theta(x\lambda + \alpha G(x))}(\lambda + \alpha g(x))$$

Substitute $F(x)$ and $\frac{\partial F}{\partial x}$ into the lemma:

$$f_{(k)}(x) = \frac{n!}{(k-1)!(n-k)!}\left[1 - e^{-\theta(x\lambda + \alpha G(x))}\right]^{k-1}\left[e^{-\theta(x\lambda + \alpha G(x))}\right]^{n-k} \theta(\lambda + \alpha g(x))(x\lambda + \alpha G(x))^{\theta-1} e^{-(x\lambda + \alpha G(x))^{\theta}}$$

$$= \frac{n!}{(k-1)!(n-k)!} e^{-\theta(x\lambda + \alpha G(x))n} \lambda = \frac{n!}{(k-1)!(n-k)!} e^{-\theta x\lambda n} e^{-\theta\alpha G(x)n} \lambda$$

$$f_{(k)}(x) = \binom{n}{k-1} e^{-n(\theta x\lambda + \theta\alpha G(x))} \lambda \tag{12}$$

4. Estimation

The likelihood function L L for a sample $X_1, X_2,..., X_n$ is the product of the individual probability density functions:

$$L(\lambda,\alpha,\theta)=\prod_{i=1}^{n} f(x_i)=\prod_{i=1}^{n}\theta(\lambda+\alpha g(x))(x\lambda+\alpha G(x))^{\theta-1}e^{-(x\lambda+\alpha G(x))^{\theta}}$$

$$\log L(\lambda,\alpha,\theta)=\sum_{i=1}^{m}\log\left(\theta(\lambda+\alpha g(x))(x\lambda+\alpha G(x))^{\theta-1}e^{-(x\lambda+\alpha G(x))^{\theta}}\right)$$

$$\log L(\lambda,\alpha,\theta)=\sum_{i=1}^{m}\left[\log\theta+\log(\lambda+\alpha g(x_i))+(\theta-1)\log(x_i\lambda+\alpha G(x_i))-(x_i\lambda+\alpha G(x_i))\theta\right]$$

Partial derivation with respect to λ

$$\frac{\partial \log L}{\partial\lambda}=\sum_{i=1}^{n}\left[\frac{\alpha\theta g(x_i)}{\lambda+\alpha g(x_i)}-x_i\theta\right]$$

$$\sum_{i=1}^{n}\left[\frac{\alpha\theta g(x_i)}{\lambda+\alpha g(x_i)}-x_i\theta\right]=0$$

Partial derivation with respect to α

$$\frac{\partial \log L}{\partial\alpha}=\sum_{i=1}^{n}\left[\frac{\theta(\lambda+\alpha g(x_i))G(x_i)}{\lambda+\alpha g(x_i)}-x_i\theta\right]$$

$$\sum_{i=1}^{n}\left[\frac{\theta(\lambda+\alpha g(x_i))G(x_i)}{\lambda+\alpha g(x_i)}-x_i\theta\right]=0$$

Partial derivation with respect to θ

$$\frac{\partial \log L}{\partial\theta}=\sum_{i=1}^{m}\left[\frac{1}{\theta}+\log(\lambda+\alpha g(x_i))+\log(x_i\lambda+\alpha G(x_i))-(x_i\lambda+\alpha G(x_i))\right]$$

$$\sum_{i=1}^{n}\left[\frac{1}{\theta}+\log(\lambda+\alpha g(x_i))+\log(x_i\lambda+\alpha G(x_i))-(x_i\lambda+\alpha G(x_i))\right]=0$$

Partial derivation with respect to λ

$$\sum_{i=1}^{n}\left[\frac{\alpha\theta g(x_i)}{\lambda+\alpha g(x_i)}-x_i\theta\right]=0$$

$$\frac{\alpha\theta g(x_i)}{\lambda+\alpha g(x_i)}=x_i\theta$$

$$\alpha\theta g(x_i) = -x_i\theta(\lambda + \alpha g(x_i))$$

$$\alpha\theta g(x_i) = x_i\theta\lambda + \alpha g x_i(x_i)$$

$$\alpha\theta g(x_i) = x_i\theta\lambda + \alpha\theta g x_i(x_i)$$

$$(1 - x_i\alpha\theta)g(x_i) = x_i\theta\lambda$$

$$g(x_i) = \frac{x_i\theta\lambda}{1 - x_i\alpha\theta}$$

Partial derivation with respect to α

$$\sum_{i=1}^{n}\left[\frac{\theta(\lambda + \alpha g(x_i))G(x_i)}{\lambda + \alpha g(x_i)} - x_i\theta\right] = 0$$

$$\sum_{i=1}^{n}\left[\frac{\theta\left(\lambda + \alpha\frac{x_i\theta\lambda}{1 - x_i\alpha\theta}\right)G(x_i)}{\lambda + \alpha\frac{x_i\theta\lambda}{1 - x_i\alpha\theta}} - x_i\theta\right] = 0$$

$$\sum_{i=1}^{n}\left[\frac{\theta(\lambda + x_i\theta\lambda\alpha)}{\lambda + x_i\theta\lambda\alpha}G(x_i) - x_i\theta\right] = 0$$

$$\sum_{i=1}^{n}\left[\frac{\theta(\lambda + x_i\theta\lambda\alpha)}{\lambda + x_i\theta\lambda\alpha}G(x_i)\right] - \sum_{i=1}^{n}x_i\theta = 0$$

Partial derivation with respect to λ

$$g(x_i) = \frac{x_i\theta\lambda}{1 - x_i\alpha\theta} = \int_{-\infty}^{x_i} g(u)\partial u = \int_{-\infty}^{x_i}\frac{u\theta\lambda}{1 - u\alpha\theta}\partial u = \frac{x_i\theta\lambda}{1 - x_i\alpha\theta}$$

Partial derivation with respect to λ

$$G(x_i) = -\frac{\lambda\log(1 - x_i\alpha\theta)}{\alpha\theta} + C$$

$$\frac{\partial G(x_i)}{\partial\lambda} = -\frac{\log(1 - x_i\alpha\theta)}{\alpha\theta}$$

Partial derivation with respect to α

$$\sum_{i=1}^{n}\left[\frac{\theta(\lambda + x_i\theta\lambda\alpha)}{\lambda + x_i\theta\lambda\alpha}\left(-\frac{\lambda\log(1 - x_i\alpha\theta)}{\alpha\theta} + C\right)\right] - \sum_{i=1}^{n}x_i\theta = 0$$

Partial derivation with respect to θ

$$\sum_{i=1}^{n}\left[\frac{1}{\theta}+\log\left(\lambda+\alpha g\left(x_i\right)\right)+\log\left(x_i\lambda+\alpha G\left(x_i\right)\right)-\left(x_i\lambda+\alpha G\left(x_i\right)\right)\right]=0$$

Solving for λ

The MLE equation for λ is derived from the partial derivative with respect to λ:

$$\frac{\partial \log L}{\partial \lambda}=\sum_{i=1}^{n}\left[\frac{\alpha\theta g\left(x_i\right)}{\lambda+\alpha g\left(x_i\right)}-x_i\theta\right]=0$$

$$\sum_{i=1}^{n}\frac{\alpha\theta g\left(x_i\right)}{\lambda+\alpha g\left(x_i\right)}=\sum_{i=1}^{n}x_i\theta$$

$$\sum_{i=1}^{n}\alpha\theta g\left(x_i\right)=\sum_{i=1}^{n}x_i\theta\left(\lambda+\alpha g\left(x_i\right)\right)$$

$$\sum_{i=1}^{n}\alpha\theta g\left(x_i\right)=\sum_{i=1}^{n}x_i\theta\lambda+\sum_{i=1}^{n}x_i\theta\alpha g\left(x_i\right)$$

$$\sum_{i=1}^{n}\alpha\theta g\left(x_i\right)-\sum_{i=1}^{n}x_i\theta\alpha g\left(x_i\right)=\sum_{i=1}^{n}x_i\theta\lambda$$

$$\sum_{i=1}^{n}\alpha\theta g\left(x_i\right)\left(1-x_i\theta\right)=\sum_{i=1}^{n}x_i\theta\lambda$$

$$\lambda=\frac{\sum_{i=1}^{n}\alpha\theta g\left(x_i\right)\left(1-x_i\theta\right)}{\sum_{i=1}^{n}x_i\theta} \tag{9}$$

Solving for α

The MLE equation for α is derived from the partial derivative with respect to α:

$$\frac{\partial \log L}{\partial \alpha}=\sum_{i=1}^{n}\left[\frac{\theta\left(\lambda+\alpha g\left(x_i\right)\right)G\left(x_i\right)}{\lambda+\alpha g\left(x_i\right)}-x_i\theta\right]=0$$

$$\sum_{i=1}^{n}\frac{\theta\left(\lambda+\alpha g\left(x_i\right)\right)G\left(x_i\right)}{\lambda+\alpha g\left(x_i\right)}=\sum_{i=1}^{n}x_i\theta$$

$$\sum_{i=1}^{n}\theta\left(\lambda+\alpha g\left(x_i\right)\right)G\left(x_i\right)=\sum_{i=1}^{n}x_i\theta\left(\lambda+\alpha g\left(x_i\right)\right)$$

$$\sum_{i=1}^{n}\theta\lambda G(x_i)+\sum_{i=1}^{n}\theta\lambda\alpha g(x_i)G(x_i)=\sum_{i=1}^{n}x_i\theta\lambda+\sum_{i=1}^{n}x_i\theta\alpha g(x_i)$$

$$\sum_{i=1}^{n}\theta\alpha g(x_i)G(x_i)=\sum_{i=1}^{n}x_i\theta\alpha g(x_i)$$

$$\alpha\sum_{i=1}^{n}\theta g(x_i)G(x_i)=\alpha\sum_{i=1}^{n}x_i\theta g(x_i)$$

$$\sum_{i=1}^{n}g(x_i)G(x_i)=\sum_{i=1}^{n}x_i g(x_i)$$

$$\alpha=\frac{\sum_{i=1}^{n}x_i g(x_i)}{\sum_{i=1}^{n}g(x_i)G(x_i)} \tag{10}$$

To solve for θ using the lemma, we will start with the MLE equation for θ derived from the partial derivative with respect to θ :

$$\frac{\partial \log L}{\partial\theta}=\sum_{i=1}^{n}\left[\frac{1}{\theta}+\log(\lambda+\alpha g(x_i))+\log(x_i\lambda+\alpha G(x_i))-(x_i\lambda+\alpha G(x_i))\right]=0$$

$$\sum_{i=1}^{n}\left[\frac{1}{\theta}+\log\left(\lambda+\alpha\frac{x_i\theta\lambda}{1-x_i\alpha\theta}\right)+\log\left(x_i\lambda+\alpha\left(-\frac{\lambda\log(1-x_i\alpha\theta)}{\alpha\theta}+C\right)\right)-\left(x_i\lambda+\alpha\left(-\frac{\lambda\log(1-x_i\alpha\theta)}{\alpha\theta}+C\right)\right)\right]=0$$

$$\sum_{i=1}^{n}\left[\frac{1}{\theta}+\log(\lambda+\alpha g(x_i))+\log(x_i\lambda+\alpha G(x_i))-(x_i\lambda+\alpha G(x_i))\right]=0 \tag{11}$$

It may be noted here that equations α cannot be solved analytically. It is estimated using R package

5. Random Number Generation

Calculate the Cumulative Distribution Function (CDF): Integrate the density function $f(x)$ to obtain the CDF $F(x)$

Find the Inverse CDF: Solve for in terms of the uniform random variable between 0 and 1. This gives the inverse CDF, denoted as $F^{-1}(U)$.

Generate Uniform Random Numbers: Generate n random numbers $u_1,u_2,\ldots,u_n$ from a uniform distribution between 0 and 1

$$F(x)=1-e^{-\theta(x\lambda+\alpha G(x))}$$

$$U=1-e^{-\theta(x\lambda+\alpha G(x))}$$

$$e^{-\theta(x\lambda+\alpha G(x))} = 1-U$$

$$-\theta(x\lambda+\alpha G(x)) = In(1-U)$$

$$x\lambda+\alpha G(x) = -\frac{1}{\theta} In(1-U)$$

$$x = -\frac{\alpha}{\lambda} G(x) - \frac{1}{\theta\lambda} In(1-U) \tag{13}$$

6. Simulation procedure

The simulation procedure aims to assess the behavior of AYUF distribution parameters under various scenarios through 1000 replications in R statistical software. Each replication involves the following steps:

Data Generation: Generate a random sample of size 100 from the AYUF distribution using the inverse transform method.

Parameter Variation: Investigate three parameter sets: (1, 0, 1), (1, 1, 1), and (1, 2, 1) for λ, α, θ, respectively

Random Number Generation: To generate random numbers from the AYUF distribution, we use the equation $x = -\frac{\alpha}{\lambda} G(x) - \frac{1}{\theta\lambda} In(1-U)$, where U is a random number uniformly distributed between 0 and 1. This equation is applied to each U to obtain the corresponding x

Maximum Likelihood Estimation (MLE): Compute the maximum likelihood estimates (MLEs) for the parameters based on the generated dataset.

Bias and MSE Calculation: Calculate the bias and Mean Square Error (MSE) of the MLEs to assess estimation accuracy.

Analysis: Analyze the results to understand how different parameter configurations impact estimation accuracy and identify any trends or patterns in estimation performance. The result is presented in the table 1 below:

Table 1 Simulation Result

	True Parameters			**Estimated paramaters**								
Distriutions	Lambda	Alpha	Theta	lambda	MSE	Bias	alpha	MSE	Bias	theta	MSE	Bias
Ayuf-Exponential	1	0	1	1.6221	0.00888	0.0437	0.0152	0.00054	-0.0152	0.8792	0.0041	-0.0264
	1	1	1	2.5331	0.0125	0.0345	0.9173	0.00987	0.01168	0.8919	0.0091	-0.0138
	1	2	1	1.112	41.1159	0.9659	1.2483	37.618	-0.563	2.4043	86.78	-3.31
Ayuf-Gamma	1	0	1	1.8192	1.2897	0.8192	1.3405	3.2775	-1.3405	0.8206	0.0705	-0.1794
	1	1	1	2.5116	8.9429	1.5116	0.9237	15.4444	-1.9237	0.9228	0.0572	-0.0772
	1	2	1	1.0805	0.7628	0.0805	1.9213	1.075	-0.0787	1.0233	0.0146	0.0233
Ayuf-Weibull	1	0	1	1.64756	1.16802	0.7419	1.214	2.96827	-1.214	0.7432	0.0638	-0.1625
	1	1	1	2.27464	8.09916	1.369	0.8366	13.9873	-1.7422	0.8357	0.0518	-0.0699
	1	2	1	0.97856	0.69083	0.0729	1.74	0.97358	-0.0713	0.9268	0.0132	0.0211

Ayuf-Inverse Weibull	1	0	1	1.0033	0.0178	0.0033	0.0199	0.159	-0.0199	0.9979	0.0053	-0.0021
	1	1	1	1.0422	0.046	0.0422	0.9814	0.6347	-0.0186	1.028	0.0163	0.028
	1	2	1	1.1024	0.0666	0.1024	1.8457	1.0532	-0.1543	1.0417	0.0145	0.0417
Ayuf-Inverse Exponential	1	0	1	0.90864	0.01612	0.003	0.018	0.144	-0.018	0.9038	0.0048	-0.0019
	1	1	1	0.94387	0.04166	0.0382	0.8888	0.57482	-0.0168	0.931	0.0148	0.02536
	1	2	1	0.99839	0.06032	0.0927	1.6716	0.95383	-0.1397	0.9434	0.0131	0.03777

The simulation results showcase the performance of parameter estimation for the GAYUF d under different scenarios and baseline distributions. Overall, the estimated parameters generally exhibit close proximity to the true values for λ and θ, with moderate bias and MSE. However, estimation accuracy varies considerably for α, especially in scenarios with higher α values, leading to higher bias and MSE. This suggests that estimation of α may be more challenging compared to λ and θ, particularly when α is larger.

7. Real Life Application

In order to evaluate the suitability of GAYUF, we have examined a set of 63 actual observations pertaining to the tensile strength of 1.5 cm glass fibres. The first dataset was acquired by employees at the UK National Physical Laboratory and utilised by Smith and Naylor (1987), whereas the second dataset focused on Chloride Data. The data set utilised by Bhaumik et al., 2009 consists of vinyl chloride measurements in milligrammes per litre (mg/l) taken from monitoring wells located upstream of the contamination source. In addition, we have conducted a comparison between the suggested GAYUF model and five different lifespan distributions: exponential, inverse exponential, gamma, Weibull, and inverse Weibull. In order to estimate the parameters, we employed the greatest likelihood technique. Based on the information provided in this table, we may draw the following conclusions:

For the data set 1, Comparing GAYUF-Weibull to the conventional Weibull model, GAYUF shows superior fit with lower AIC (32.723 vs. 34.41368) and BIC (32.12102 vs. 645.7463) values. Similarly, GAYUF Gamma surpasses the conventional Gamma model with lower AIC and BIC, indicating better fit. Additionally, GAYUF-Inverse Weibull outperforms the standard Inverse Weibull with substantially lower AIC and BIC values, suggesting improved goodness of fit

For data set 2, When comparing GAYUF-Exponential to the conventional Exponential model, Ayuf-Exponential exhibits a slightly lower AIC (115.11 vs. 112.91) and BIC (113.7044 vs. 115.0531), suggesting a slightly better fit. Similarly, GAYUF-Weibull surpasses the conventional Weibull model with lower AIC and BIC values (116.7739 vs. 114.9 and 115.3683 vs. 119.1863, respectively), indicating improved goodness of fit. Additionally, GAYUF-Gamma outperforms the standard Gamma model with lower AIC and BIC, indicating better fit.

8. Conclusion

In the present piece of work, we have proposed a new lifetime family of distribution capable of handling various forms of hazard rate. Various statistical properties were obtained. We conducted a simulation study, it is evident that the proposed GAYUF distribution is a flexible model for application. The distribution may be used as a lifetime model and can be fitted on the life length of various components. Additionally, we have considered two different real data sets and five other distributions. It is shown that the few of the considered distributions may fit some of the data, but GAYUF generated all considered the two data sets very well even better than the other distributions which are providing a good fit to the specific data. Thus one can easily conclude that our proposed (GAYUF) distributions is more flexible and can be considered as a suitable model for a large variety of lifetime data.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to be disclosed.

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Appendix

Table 2 Summary Statistics of Real Life Datasets

Statistic	Dataset1	Dataset2
mean	1.5068	1.8794
sd	0.32413	1.9526
kurtosis	3.9238	5.0054
skewness	-0.89993	1.6037

median	1.59	1.15
mode	1.61	0.4

Dataset 1

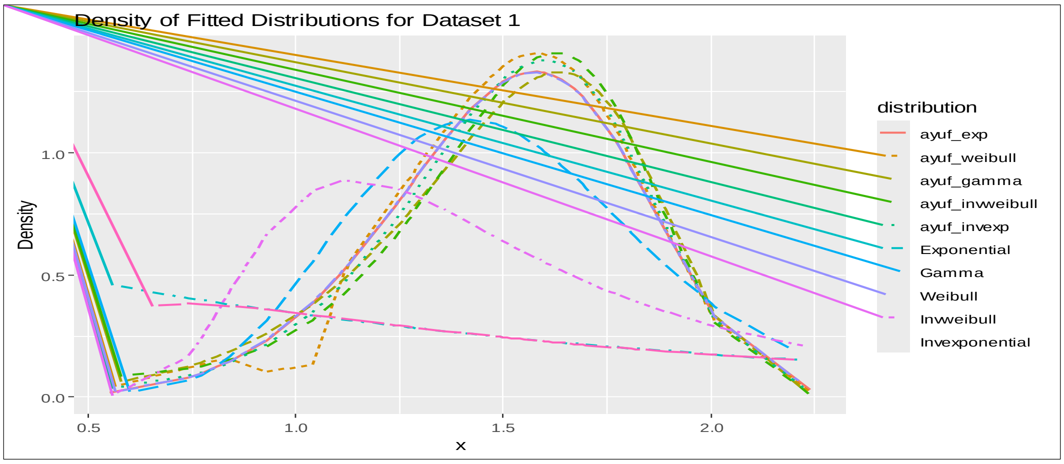

Table 3 MLE, AIC, BIC and K-S statistics with p-value for fitted data sets.

Model	Parameter	Estimate	Std.	-2log(l)	AIC	BIC	K-S
Ayuf-Exponential	lambda	0.737244	0.407679	30.31576	36.31576	35.71378	0.135808
	alpha	-0.24932	0.824326				
	theta	5.170387	1.892839				
Ayuf-Weibull	lambda	0.684227	0.0313	26.723	32.723	32.12102	0.12128
	alpha	-0.11831	0.045046				
	theta	5.484789	0.547739				
Ayuf-Gamma	lambda	1.198758	0.400128	28.48927	34.48927	33.88729	0.118911
	alpha	-1.98665	1.35752				
	theta	6.354736	0.706131				
Ayuf-Inverse Weibull	lambda	0.859406	0.122362	26.76864	32.76864	32.16666	0.106441
	alpha	-0.5993	0.29533				
	theta	5.665097	0.54018				
Ayuf-Inverse Exponential	lambda	1.00867	0.303988	28.6838	34.6838	34.08182	0.119982
	alpha	-1.19855	0.919779				
	theta	4.841839	0.763566				
Exponential	lambda	0.663647	0.083611	177.6606	179.6606	662.1831	0.402124
Gamma	shape	17.43947	3.078004	47.90308	51.90308	642.8863	0.200516
	rate	11.5737	2.072344				
Weibull	shape	5.780708	0.576095	30.41368	34.41368	645.7463	0.13636
	scale	1.628115	0.037094				
Inverse Wibul	shape	2.887556	0.234431	93.70663	97.70663	677.0463	0.228562
	rate	0.790926	0.036817				

Inverse Exponential	lambda	0.710028	0.089455	178.8784	180.8784	677.2631	0.487886

Case 2: Chloride Data

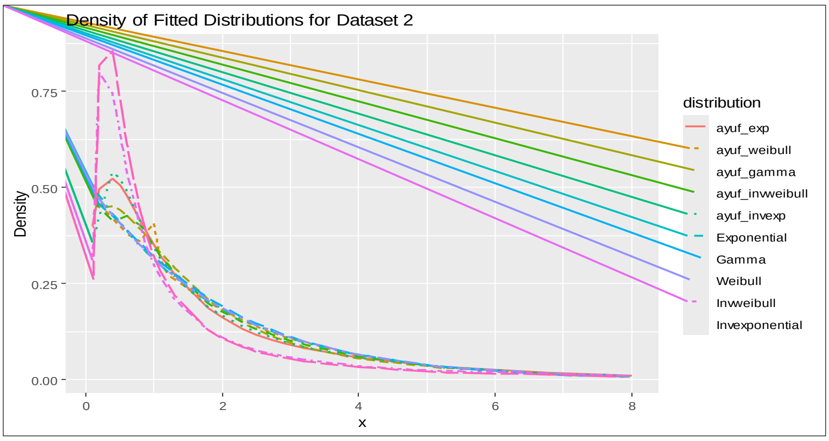

Table 4 MLE, AIC, BIC and K-S statistics with p-value for fitted data sets.

Model	Parameter	Estimate	Std.	-2log(l)	AIC	BIC	K-S
Ayuf-Exponential	lambda	0.215475	0.125101	109.11	115.11	113.7044	0.075234
	alpha	0.770066	0.340244				
	theta	1.515822	0.377588				
Ayuf-Weibull	lambda	0.517997	0.099036	110.7739	116.7739	115.3683	0.085554
	alpha	0.034624	0.104062				
	theta	1.013831	0.13399				
Ayuf-Gamma	lambda	0.422244	0.177001	110.4801	116.4801	115.0745	0.089604
	alpha	0.482844	0.729385				
	theta	1.018302	0.136634				
Ayuf-Inverse Weibull	lambda	0.496033	0.122254	110.7334	116.7334	115.3278	0.091496
	alpha	0.138931	0.348182				
	theta	1.008872	0.133884				
Ayuf-Inverse Exponential	lambda	0.40739	0.134656	109.8238	115.82	114.4182	0.076768
	alpha	0.514637	0.497923				
	theta	1.05164	0.146465				
Exponential	lambda	0.53208	0.091251	110.91	112.91	115.0531	0.088956
Gamma	shape	1.06265	0.22814	110.83	114.83	119.1163	0.097329
	rate	0.56543	0.15357				
Weibull	shape	1.0102	0.13266	110.9	114.9	119.1863	0.091841
	scale	1.8879	0.33904				
Inverse Wibul	shape	0.88041	0.10933	117.25	121.25	125.5363	0.099517
	rate	1.61998	0.3352				
Inverse Exponential	lambda	1.7468	0.29956	118.39	120.39	122.5331	0.11759