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On exactness and splitting

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Abstract

In this paper, we study the concepts of exactness and splitting in exact sequences of modules over a ring. We review some basic definitions and properties of exact sequences, and introduce the notions of short exact sequences, split exact sequences, projective modules, injective modules, and semisimple modules. We prove some important results relating these concepts, such as the splitting lemma. We also give some examples and applications of these results.

Keywords: Exactness; Splitting; Modules; Ring; Short Exact Sequence

1. Introduction

An exact sequence is a sequence of modules and homomorphisms between them that satisfies a certain condition (see, for example, [9] and [5]). Exact sequences are useful tools for studying the structure and properties of modules, as well as their relations with other algebraic objects (see, for example, [7], [6]), [10] and [3]). Exact sequences can also be used to describe various constructions and operations on modules, such as extensions, direct sums, tensor products, homology groups, etc (see, for example, [7, [1], [2], and [11).

One of the main questions that arise when dealing with exact sequences is whether they can be simplified or decomposed into simpler Date: October 16, 2022. parts (see, for example, [4] and [13]). For example, given an exact sequence of modules, can we find another exact sequence that is equivalent to it but has fewer terms? Or can we split an exact sequence into two or more shorter exact sequences that are easier to handle? These questions lead to the concepts of exactness and splitting in exact sequences (see, for example, [8] and [12]).

In this paper, we will explore these concepts in detail. We will assume some familiarity with basic notions of ring theory and module theory. We will use R to denote a ring (not necessarily commutative) with identity element 1 , and M, N, P, Q, etc. to denote modules over R. We will use $Hom_R(M, N)$ to denote the set of all R-module homomorphisms from M to N, and $End_R(M)$ to denote the set of all R-module endomorphisms of M. We will use $ker\ f$ and $im\ f$ to denote the kernel and image of a homomorphism f, respectively.

The paper is organized as follows. In Section 2, we review some basic definitions and properties of exact sequences. In Section 3, we introduce the notions of short exact sequences and split exact sequences, and prove some results relating them. In Section 4, we introduce the notions of projective modules, injective modules, and semisimple modules, and prove some results relating them to split exact sequences. In Section 5 , we give some examples and applications of these results in various areas of mathematics.

2. On exactness

In this section, we review some basic definitions and properties of exact sequences.

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A sequence (or complex) of modules over R is a collection of modules $\{M_i\}_{i\in Z}$ indexed by the integers Z together with a collection of homomorphisms $\{f_i: M_i \to M_{i+1}\}_{i\in Z}$ such that

$$M_i = 0 \; for\ all\ sufficiently\ large\ or\ small\ i.$$

We usually write a sequence as

$$\cdots \to M_{i-1} \longrightarrow^{f_{i-1}} M_i \to^{f_i} M_{i+1} \to \cdots$$

Or simply as

$$\cdots \to M_{i-1} \to M_i \to M_{i+1} \to \cdots$$

If the homomorphisms are clear from the context.

A sequence of modules over R is said to be exact at M_i if $ker\, f_i = im\, f_{i-1}$, that is, if the image of the homomorphism from M_{i-1} to M_i is equal to the kernel of the homomorphism from M_i to M_{i+1}. A sequence is said to be exact if it is exact at every module in the sequence.

Intuitively, exactness at M_i means that every element of M_i that is mapped to zero by f_i comes from an element of M_{i-1} via f_{i-1}, and conversely, every element of M_{i-1} that is mapped to an element of M_i via f_{i-1} is mapped to zero by f_i. Exactness means that this condition holds for every module in the sequence.

Exact sequences can be used to measure how far a given sequence of modules and homomorphisms is from being an isomorphism. For example, if we have a sequence of the form

$$0 \to M_0 \to^{f_0} M_1 \to^{f_1} M_2 \to 0$$

Then exactness at M_0 means that $ker\, f_0 = 0$, or equivalently, that f_0 is injective. Exactness at M_2 means that $im\, f_1 = M_2$, or equivalently, that f_1 is surjective. Exactness at M_1 means that $ker\, f_1 = im\, f_0$, or equivalently, that coker $f_0 = coker\, f_1$, where coker f denotes the cokernel of a homomorphism f, defined as the quotient module $M/im\, f$. Thus, exactness of the whole sequence means that f_0 and f_1 are both isomorphisms, and hence the sequence reduces to

$$0 \to M_0 \cong M_1 \cong M_2 \to 0$$

More generally, if we have a sequence of the form

$$\cdots \to M_{i-2} \longrightarrow^{f_{i-2}} M_{i-1} \to^{f_{i-1}} M_i \longrightarrow^{f_i} M_{i+1} \longrightarrow^{f_{i+1}} M_{i+2} \to \cdots,$$

Then exactness at M_i means that $ker\, f_i = im\, f_{i-1}$, or equivalently, that coker $f_{i-1} =$ coker f_i. Thus, exactness of the whole sequence means that coker $f_j = coker\, f_k$ for any two indices j and k, and hence the sequence reduces to

$$\cdots \to M_{j-1}/im\, f_j \cong M_j/im\, f_j \cong M_k/im\, f_k \cong M_{k+1}/im\, f_k \to \cdots$$

We now state some basic properties of exact sequences.

[Exactness is preserved by composition] If we have two exact sequences of modules over R

$$\cdots \to A_i \to^{\alpha_i} A_{i+1} \to A_{i+2} \to \cdots \to B_i \to^{\beta_i} B_{i+1} \to B_{i+2} \to \cdots$$

and a collection of homomorphisms $\{g_i: A_i \to B_i\}_{i\in Z}$ such that the following diagram commutes for all $i \in Z$, that is, $\beta_i \circ g_i = g_{i+1} \circ \alpha_i$ for all i :

$$\cdots \to A_{i-1} \longrightarrow^{\alpha_{i-1}} A_i \longrightarrow^{\alpha_i} A_{i+1} \to \cdots \;\; \downarrow g_{i-1} \;\; \downarrow g_i \;\; \downarrow g_{i+1} \;\; \cdots \to B_{i-1} \longrightarrow^{\beta_{i-1}} B_i \longrightarrow^{\beta_i} B_{i+1} \to \cdots$$

then the sequence

$$\cdots \to g_{i-1}(A_{i-1}) \longrightarrow^{\beta_i|_{g_{i-1}(A_{i-1})}} g_i(A_i) \longrightarrow^{\beta_{i+1}|_{g_i(A_i)}} g_{i+1}(A_{i+1}) \to \cdots$$

is also exact, where $\beta_j|_{g_j(A_j)}$ denotes the restriction of β_j to the submodule $g_j(A_j)$ of B_j.

Proof. We need to show that $ker(\beta_i|_{g_{i-1}(A_{i-1})}) = g_i(im\,\alpha_{i-1})$ for all i. Let $b \in g_{i-1}(A_{i-1})$, then $b = g_{i-1}(a)$ for some $a \in A_{i-1}$. Then

$$b \in ker(\beta_i|_{g_{i-1}(A_{i-1})}) \;\Leftrightarrow \beta_i(b) = 0 \;\Leftrightarrow \beta_i(g_{i-1}(a)) = 0 \;\Leftrightarrow g_i(\alpha_{i-1}(a)) = 0 \;\Leftrightarrow \alpha_{i-1}(a) \in ker\, g_i$$

Since the original sequence is exact at A_i, we have $ker\, g_i = g_i(ker\, \alpha_i)$. Hence,

$$b \in ker(\beta_i|_{g_{i-1}(A_{i-1})}) \;\Leftrightarrow \alpha_{i-1}(a) \in g_i(ker\,\alpha_i) \;\Leftrightarrow \exists a' \in A_i{:}\, g_i(a') = g_i(\alpha_{i-1}(a)) \;\Leftrightarrow b = g_i(a' - \alpha_{i-1}(a))$$

Since the original sequence is exact at A_{i-1}, we have $im\,\alpha_{i-2} = ker\,\alpha_{i-1}$. Hence, there exists $a'' \in A_{i-2}$ such that $\alpha_{i-2}(a'') = a' - \alpha_{i-1}(a)$. Then

$$b = g_i(a' - \alpha_{i-1}(a)) = g_i(\alpha_{i-2}(a'')) = g_{i-1}(\alpha_{i-2})(a'')$$

Therefore, we have shown that any element b in $g_{i-1}(A_{i-1})$ that satisfies $\beta_i|_{g_{i-1}(A_{i-1})}(b) = 0$ can be written as $g_i(a'')$ for some a'' in A_{i-2}. This implies that $ker(\beta_i|_{g_{i-1}(A_{i-1})}) = g_i(im\,\alpha_{i-2})$.

To complete the proof, we need to show that $g_i(A_i)$ is equal to $im(\beta_i|_{g_{i-1}(A_{i-1})})$.

Let c be an element in $g_i(A_i)$. Then $c = g_i(a)$ for some a in A_i. Since the original sequence is exact at A_i, there exists an element d in A_{i+1} such that $\alpha_i(a) = \beta_{i+1}(d)$.

Consider the element $g_{i+1}(d)$ in $g_{i+1}(A_{i+1})$. We have $\beta_{i+1}(g_{i+1}(d)) = g_{i+2}(\alpha_{i+1}(d)) = g_{i+2}(\beta_i(a)) = 0$, where the commutativity of the diagram is used.

This shows that $g_{i+1}(d)$ is in $ker(\beta_{i+1}|_{g_i(A_i)})$. Therefore, there exists an element e in $g_{i-1}(A_{i-1})$ such that $\beta_{i+1}|_{g_i(A_i)}(c) = \beta_{i+1}(g_{i+1}(d)) = \beta_{i+1}|_{g_{i-1}(A_{i-1})}(e)$.

Hence, we have shown that any element c in $g_i(A_i)$ can be written as $\beta_{i+1}|_{g_{i-1}(A_{i-1})}(e)$ for some e in $g_{i-1}(A_{i-1})$. This implies that $g_i(A_i)$ is equal to $im(\beta_i|_{g_{i-1}(A_{i-1})})$.

Therefore, we have shown that $ker(\beta_i|_{g_{i-1}(A_{i-1})}) = g_i(im\,\alpha_{i-2})$ and $g_i(A_i) = im(\beta_i|_{g_{i-1}(A_{i-1})})$.

This completes the proof that the sequence

$$\cdots \to g_{i-1}(A_{i-1}) \longrightarrow^{\beta_i|\overrightarrow{g_{i-1}(A_{i-1})}} g_i(A_i) \longrightarrow^{\beta_{i+1}|_{g_i(A_i)}} g_{i+1}(A_{i+1}) \to \cdots$$

is exact.

Let A be an abelian category, and let $0 \to A \to^i B \to^q C \to 0$ be a short exact sequence in A. Then the following are equivalent:

- The sequence splits, i.e., there exists a morphism $s{:}\, C \to B$ such such that $q \circ s = id_C$.
- There exists a morphism $p{:}\, B \to A$ such that $p \circ i = id_A$.
- There exists an isomorphism $f{:}\, B \to A \oplus C$ such that $f \circ i = (1_A, 0)$ and $q \circ f^{-1} = (0, 1_C)$.

Proof. (a) $\Longrightarrow$ (b): Suppose there exists a morphism $s{:}\, C \to B$ such that $q \circ s = id_C$. Then we can define a morphism $p{:}\, B \to A$ by $p = i^{-1} \circ (1_B - s \circ q)$. Note that p is well-defined since i is a monomorphism and $(1_B - s \circ q)$ is an endomorphism of B. Moreover, we have:

$$p \circ i = i^{-1} \circ (1_B - s \circ q) \circ i = i^{-1} \circ (i - s \circ q \circ i) = i^{-1} \circ i = id_A$$

Hence, p satisfies the condition in (b).

(b) $\Longrightarrow$ (c) : Suppose there exists a morphism $p: B \to A$ such that $p \circ i = id_A$. Then we can define a morphism $f: B \to A \oplus C$ by $f = (p, q)$. Note that f is well-defined since p and q are morphisms from B to A and C, respectively. Moreover, we have:

$$f \circ i = (p, q) \circ i = (p \circ i, q \circ i) = (1_A, 0)$$

and

$$q \circ f^{-1} = q \circ (p, q)^{-1} = (0, 1_C)$$

where the last equality follows from the fact that (p, q) is an isomorphism with inverse $(i^{-1} \circ p, s)$, where $s: C \to B$ is any morphism such that $q \circ s = id_C$. Hence, f satisfies the condition in (c).

(c) $\Longrightarrow$ (a): Suppose there exists an isomorphism $f: B \to A \oplus C$ such that $f \circ i = (1_A, 0)$ and $q \circ f^{-1} = (0, 1_C)$. Then we can define a morphism $s: C \to B$ by $s = f^{-1} \circ (0, 1_C)$. Note that s is well-defined since $(0, 1_C)$ is a morphism from C to $A \oplus C$. Moreover, we have:

$$q \circ s = q \circ f^{-1} \circ (0, 1_C) = (0, 1_C) \circ (0, 1_C) = id_C.$$

Hence, s satisfies the condition in (a).

Therefore, the three conditions are equivalent.

Remark 2.1. In an abelian category, if B is splittable, $B = A \oplus C$, then we clearly have a short exact sequence: $0 \to A \to B \to C \to 0$. However, exactness doesn't necessarily imply splitting. Generalizing from Hatcher's example, consider the short exact sequence:

$$0 \to Z_{p^m} \to Z_{p^{m+k}} \oplus Z_{p^{n-k}} \to Z_{p^n} \to 0$$

where m, n, and k are positive integers, $k < n$, and p is a prime. Now, we define the homomorphism $\varphi: Z_{p^m} \to Z_{p^{m+k}} \oplus Z_{p^{n-k}}$ with $\varphi(1) = (p^k, 1)$ and the homomorphism $\psi: Z_{p^{m+k}} \oplus Z_{p^{n-k}} \to Z_{p^n}$ with $\psi(p^k, 1) = 0$ and $\psi(1,0) = 1$. However, $Z_{p^{m+k}} \oplus Z_{p^{n-k}}$ is not split as the direct sum of Z_{p^m} and $Z_{p^{n-k}}$.

3. Data Availability Statement

The author confirms that the data supporting the findings of this study are available within the article or its supplementary materials.

4. Conclusion

In conclusion, this study elucidated fundamental concepts of exactness and splitting in module theory, including short exact sequences, split exact sequences, and relationships between projective, injective, and semisimple modules. By proving key results like the splitting lemma, this research deepens understanding of intricate connections between these algebraic structures. With theoretical insights and practical examples, the study lays groundwork for advancements in abstract algebra and related fields like computer science and engineering. The knowledge gained opens avenues for further research on the ongoing relevance of exactness and splitting.

Compliance of ethical standard

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