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(Research Article)

Mathematical analysis and implications of new definition of singular integral operator

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Abstract

Let D be a connected bounded domain in R2, S be its boundary which is closed, connected and smooth or

$$S = (-\infty, \infty). \text{ Let } \Phi(z) = \frac{1}{2\pi i} \int_S$$

$$i\pi \quad S$$

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1. Introduction

Let D be a connected bounded domain on the complex plane, S be its boundary, which is closed and C1,a-smooth, 0 < a ≤ 1 or S = (−∞, ∞). The standard definition of the singular integral operator

nooth, $0 < a \leq 1$ o

$$Af = \frac{1}{i\pi} \int_S \frac{f(s)ds}{s-t} \text{ is:}$$

……………..(1)

We assume that f L1(S). This is the basic new assumption: in the literature it was assumed that f Hμ(S), where Hμ(S) is the space of Ho¨lder-continuous functions, or f Lp(S), p > 1, see [2], [4]. In [1] there is a result for f L1(S), the existence of the limit (1) is proved, and the proof is far from simple. Our goal is to give a new definition of the operator A. This definition makes the proof of the existence of Af for f L1(S) very simple. It is also of great interest to have a proof of the Sokhotsky formulas for f L1(S), see also [6].

Definition 1

$$(Af, \varphi) = (f, A\varphi) \quad \forall \varphi \in H\mu(S),\ 0 < \mu < 1 \ldots\ldots\ldots\ldots\ldots \qquad (2)$$

Here (Af, φ) = 1 ∫

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Lemma 1

Formula (2) defines f ∈ L1(S) uniquely.

Proof. Suppose that f1, f2 ∈ L1(S) satisfy (2). Then q := f1 − f2 satisfies the relation

(q, Aφ) = 0 for all φ ∈ Hμ(S). It is known [2] that the set Aφ|∀φ∈Hμ(S) = Hμ(S) if 0 < μ < 1.

Therefore, q ∈ L1(S) is orthogonal to the set Hμ(S) dense in L1(S). Consequently, q = 0 and

f1 = f2. Lemma 1 is proved. Q

Let us check that the right side of formula (2) makes sense. This mathematical expression can be written as

dsdtf (s)φ(t) 1 . The integrand here is absolutely integrable over S S. Therefore, the order of integration can be changed and formula (2) makes sense.

There are other advantages of Definition 1. For example, it is easy to prove that the operator A is closed.

Lemma 2

The operator A in L1(S) is closed.

Proof. One has to prove that the graph f, Af is a closed set in L1(S) L1(S). Let fn f and Afn h, convergence in L1(S). Then, by Definition 1, (fn, Aφ) (f, Aφ) = (Af, φ) and (Afn, φ) (h, φ). Therefore, (Af h, φ) = 0 φ Hμ(S). Since Hμ(S) is dense in L1(S), it follows that Af = h. Thus, A is closed. Q

However, A is not continuous in L1(S).

Example 1

Let us sh∫ow that there is an f ∈ L1(S) such that Af ∈/ L1(S).

known formula, see [3]: F(s^{-1}) = iπsgn(ξ), where sgn(ξ) = 1 if ξ > 0, sgn(ξ) = −1 if ξ < 0, sgn(0) = 0. The Fourier transform of f ∈ L1(S) is a continuous uniformly bounded function. Therefore, f˜sgn(ξ) is not, in general, a continuous function at ξ = 0. Thus, if f˜|ξ=0 /= 0, then the function Af ∈/ L1(S).

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2. Other results

- Consider the equation Af = h, h L1(S). From Example 1 it follows that a necessary and sufficient condition for the sovability of this equation in L1(S) is the condition h˜ = 0. If this equation is solvable, then its solution is unique. Indeed, if f1 and f2 are solutions, then q = f1 − f2 solves the equation Aq = 0. Taking its Fourier transform leads to the relation q˜sgn(ξ) = 0. Therefore, q˜ = 0 for ξ /= 0. Since q ∈ L1(S), it follows that q˜ = 0 so q = 0 and f1 = f2. Q
- Let us prove the generalization of the Sokhotsky-Plemelj formulas to the case when

$f \in L^1(S)$. Let $\Phi(z) = \frac{1}{2\pi i}\int_S \frac{f(s)ds}{s-z}$. Then

$$\Phi(z) = f(t)v(z) + \psi(z), \quad \psi(z) := c\int_S \frac{f(s)-f(t)}{s-z}ds, \quad t \in S, \quad v(z) := c\int_S \frac{ds}{s-z}, \tag{3}$$

and

$$v(z) = \begin{cases} 1, & z \in D, \\ \frac{1}{2}, & z \in S, \\ 0, & z \in D'. \end{cases} \tag{4}$$

One has

$$\Phi^{+}(t) = \lim_{z \to t, z \in D} \Phi(z) = f(t) + \psi^{+}(t) \tag{5}$$

where $\psi^{+}(t) = \lim_{z \to t, z \in D} \psi(z)$ and

$$\Phi^{-}(t) = \lim_{z \to t, z \in D'} \Phi(z) = \psi^{-}(t). \qquad (6)$$

If $t \in S$, then one gets (see equation (4), the line $z \in S$):

$$\Phi(t) = \frac{f(t)}{2} + \psi(t) := \frac{f(t)}{2} + c \int_S \frac{f(s) - f(t)}{s - t} ds. \qquad (7)$$

The ψ(t) is the value of ψ(z) at z = t. The ψ(t) and Φ(t) are understood as in Definition 1

If some equation holds almost everywhere with respect to the Lebesgue measure on S, then we write that this equation holds a.e.

From formulas (3)–(6) one derives:

$\Phi+(t) - \Phi-(t) = f(t) + \psi+(t) - \psi-(t)$ a.e., $\Phi+(t) + \Phi-(t) = f(t) + \psi+(t) + \psi-(t)$ a.e ……………. (8)

In Lemma 3 we prove that ψ+(t) = ψ−(t) = ψ(t) a.e. Therefore, formula (8) can be rewritten as:

$\Phi+(t) - \Phi-(t) = f(t)$ a.e., $\Phi+(t) + \Phi-(t) = f(t) + 2\psi(t)$ a.e…………….(9)

From equation (9) it follows that

$$\Phi_{+}(t) = \Phi(t) + \frac{f(t)}{2}, \quad \Phi_{-}(t) = \Phi(t) - \frac{f(t)}{2}, \text{ a.e.} \qquad (10)$$

where Φ(t) = ψ(t). Formulas (10) are the Sokhotsky-Plemelj formulas for f L1(S).

Theorem 1

For f L1(S) formulas (10) hold.

To finish the proof of Theorem 1 it is sufficient to prove Lemma 3.

Lemma 3. If $f \in L1(S)$ and S is C1,a−smooth, $0 < a \leq 1$, then

$\psi+(t) = \psi-(t) = \psi(t)$ a.e…………..(11)

Before proving Lemma 3 we prove Lemma 4.

Lemma 4

One has

$$\lim_{\epsilon \to 0} \frac{1}{s - t \pm iN_t\epsilon} = \frac{1}{s - t} + i\pi\delta(s - t). \qquad (12)$$

Proof. Formula (12) is understood according to Definition 1. Let f ∈ L1(S), φ ∈ Hμ(S), Nt be a unit normal to S directed inside D. Then Furthermore,

$$\lim_{\epsilon\to 0}\int_S dt\varphi(t)\int_S\frac{f(s)ds}{s-t-iN_t\epsilon}:=\int_S dt\varphi(t)\int_S\frac{f(s)ds}{s-t-i0}.$$

Furthermore,

$$\int_S dt\varphi(t)\int_S\frac{f(s)ds}{s-t-i0}=\int_S dt\varphi(t)\int_S\frac{f(s)ds}{s-t}+i\pi\int_S\varphi(t)f(t)dt,\qquad\forall\varphi\in H^\mu(S),\tag{13}$$

and

$$\int_S dt\varphi(t)\int_S\frac{f(s)ds}{s-t}=\int_S dsf(s)\int_S\frac{\varphi(t)dt}{s-t}\qquad\forall\varphi\in H^\mu(S).\tag{14}$$

We have proved formula (12) according to Definition 1 with the minus sign. Similarly one proves this formula with the plus sign. Lemma 4 is proved.   Q In [3], p. 83, there is a formula 1 = 1 + iπδ(x) understood in the sense of distributions.

x−i0   x

The formula in Lemma 4 is of the similar type. The Sokhotsky-Plemelj formulas (10) were

derived in [2] and [5] under the assumption that f Hμ(S). Under such an assumption, these formulas hold everywhere, not almost everywhere.

Proof of Lemma 3

By Definition 1 one has (neglecting 1 and denoting by integration

over S × S):

$$\lim_{\epsilon\to 0}\int\frac{[f(s)-f(t)]\varphi(t)}{s-t\pm iN_t\epsilon}dsdt=\int\frac{[f(s)-f(t)]\varphi(t)}{s-t}dsdt\mp i\pi J=(\psi,\varphi)\tag{15}$$

where

$$J:=\int[f(s)-f(t)]\varphi(t)\delta(s-t)dsdt=\int_S\int_S[f(s)-f(t)]\varphi(t)\delta(s-t)dsdt=0.\tag{16}$$

For f ∈ Hμ(S) formula (16) is trivial by the standard definition of∫the delta-function. For f ∈

for f   L1(S). Lemma 3 is proved.   Q

Let z   D. The following question is of interest:

When is the boundary value Φ+(t) of Φ(z) on S equal to f a.e.?

Equation (10)

Yields a necessary and sufficient condition for this:

Φ+(t) = f (t) iff Φ(z) = 0, z ∈ D− and f (t) = Φ(t) + f(t) , or f (t) = 1 ∫ f (s) ds   a.e.

If one wants to formulate a necessary and sufficient condition for f (s) ∈ L (S) to be aboundary value of an analytic in D− function Φ(z), Φ(   ) = 0, then an argument, similar to the above yields the following conditions:

$$f(t) = -\frac{1}{i\pi}\int_{S}\frac{f(s)}{s-t}ds \quad a.e. \qquad (17)$$

If equation (17) holds, then Φ+(t) = 0 and, consequently, Φ(z) = 0 if z ∈ D+ = D.

Remark 1

If Φ (t) = f (t) a.e., f ∈ L (S), then Af ∈ L (S), where Af := f(s) ds a.e. If −Φ−(t) = f (t) a.e., f ∈ L1(S) and Φ(∞) = 0, then Af ∈ L1(Siπ S s−t L1(S)). Since for some f ∈ one does not have Φ(z) = 0, z D−, it follows that not every f L1(S) is a boundary value of an analytic function in D.

3. Conclusion

A new definition of singular integral operator in $L^1(S)$ is given. Sokhotsky-Plemelj formulasare derived for $f \in L^1(S)$. Other results are obtained.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to disclosed.

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