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Certain properties of the multivalent class of a second order inequality

Uzoamaka Azuka Ezeafulukwe *

Mathematics Department, Faculty of Physical Sciences, University of Nigeria, Nsukka, Nigeria.

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Abstract

In this article, we define a class of multivalent function and calculate the necessary and sufficient condition for a function to be in such class. Using some relations among $\frac{f(z)}{[z^p]}$, $\frac{f'(z)}{[z^p]'}$ and $1+\frac{f''(z)}{f'(z)}$, we show that function in this family is convex.

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1. Introduction

We denote the open unit disc by

$$U = \{z \in C : |z| < 1\}.$$

The set of functions

$$f : U \to C$$

analytic in the unit disc is denoted by H(U). For $a \in R+$ and $k \in N$. We denote and

$$\mathcal{H}[a,k] = \left\{f(z) \in \mathcal{H}(\mathcal{U}) : f(z) = a + a_k z^k + a_{k+1} z^{k+1} + \cdots\right\},$$
$$\mathcal{A}_k = \left\{f(z) \in \mathcal{H}(\mathcal{U}) : f(z) = z + a_{k+1} z^{k+1} + a_{k+2} z^{k+2} \cdots\right\} \quad . (1.1)$$

$$\mathcal{A}_k(p) = \left\{ f \in \mathcal{H}(\mathcal{U}) : f(z) = z^p + \sum_{k=2}^{\infty} p^k a_k z^{k+p-1},\ p \in \mathbb{N},\ z \neq \frac{1}{p} \right\}$$

The subclass $A_k(p)$ was defined and studied by [1] and $A_k(1) = A_k$.
A function *f* in $A(p)$ is said to be *p*-valently convex in U if and only if

* Corresponding author: Uzoamaka A. Ezeafulukwe

$$\Re e\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0 \quad (1.2)$$

and we denote the class of p-valently convex functions by C(p).
Motivated by the work done by Kanas and Owa [4] with the expression

$$(1-\alpha)\frac{f(z)}{z}+\alpha f'(z)+\beta z f'' \prec 1+Mz, \quad (1.3)$$

where,

($f(z) \in A_k(1)$), $\alpha \geq 1$, $\alpha + 2\beta \geq 0$ $\alpha,\beta \in R$,

in which they found some connections between certain second-order differential subordination and some subordination of the expressions: $\frac{f(z)}{z}$, $f'(z)$ and $1+\frac{zf''(z)}{f'(z)}$. We define the class of analytic function $\mathcal{Q}^{(2)}_{(\alpha,\beta;\gamma)}(f(z),f'(z),f''(z);p)$ which is different from Kanas and Owa (1.3) expression.

The main aim of this article is to calculate the necessary and sufficient condition for a function f to be in the class $\mathcal{Q}^{(2)}_{(\alpha,\beta;\gamma)}(f(z),f'(z),f''(z);p)$ and also estimate the coefficient bound of $f \in \mathcal{Q}^{(2)}_{(\alpha,\beta;\gamma)}(f(z),f'(z),f''(z);p)$. We also used the relations among $\frac{f(z)}{[z^p]}$, $\frac{f'(z)}{[z^p]'}$ and 1+ $\frac{f''(z)}{f'(z)}$ to show that function f is in C(p).

Definition 1.1 *Let α, β be real numbers such that $\alpha \geq 1$, $0 < \beta < 1$, $\alpha > 2\beta p$ and $f \in A_k(p)$, we introduce a class of analytic function*

$$\mathcal{Q}^{(2)}_{(\alpha,\beta;\gamma)}(f(z),f'(z),f''(z);p) = \mathcal{R}e\left\{\frac{\beta p(p-1)[f(z)]''}{[z^p]''}+\frac{[\alpha+2\beta(1-p)]p[f(z)]'}{[z^p]'}+\frac{(p\beta-\alpha)(p-1)[f(z)]}{[z^p]}\right\} > \gamma, \quad (1.4)$$

where

$z \in U$, $z \neq 0$, $p \in N$, $\alpha \neq \gamma$, $0 \leq \gamma < 1$.

Theorem 1.1 *[2] $p \in H[1,n]$ if and only if there is probability measure $\{\mu\}$ on χ such that p*

$$p(z)=\int_{\chi}\frac{1+xz}{1-xz}d\mu(x), (|z|<1)$$

and $\chi = \{x: |x| = 1\}$. The correspondence between H [1, n] and the set of probability measure $\{\mu\}$ on χ given by [3] is one-to-one.

2. The necessary and sufficient condition of the functions in the Class $\mathcal{Q}^{(2)}_{(\alpha,\beta;\gamma)}(f(z),f'(z),f''(z);p)$

In this section we calculate the necessary and sufficient condition of the function $f \in \mathcal{Q}^{(2)}_{(\alpha,\beta;\gamma)}(f(z),f'(z),f''(z);p)$ and also estimate its coefficient bounds.

Theorem 2.1 *Let $f \in A(p)$ be at least twice differentiable and $f \in \mathcal{Q}^{(2)}_{(\alpha,\beta;\gamma)}(f(z),f'(z),f''(z);p)$ be as defined in (1.4).*

A function $f \in \mathcal{Q}^{(2)}_{(\alpha,\beta;\gamma)}(f(z),f'(z),f''(z);p)$ if and only if $f(z)$ can be expressed as

$$f(z)=z^p+(\alpha-\gamma)\int_{\chi}\left[\sum_{k=1}^{\infty}\frac{2x^k}{(k+1)(\alpha+k\beta)}z^{k+p}\right]d\mu(x), \quad (2.1)$$

where $\{\mu\}$ is probability measure on χ such that $\chi = \{x : |x| = 1\}$.

Proof. Let $f \in A(p)$ with $f \in \mathcal{Q}^{(2)}_{(\alpha,\beta;\gamma)}(f(z),f'(z),f''(z);p)$. We set then by (1.4),

F(z) = βz2ϕ″ (z) + [α + 2β] zϕ′ (z) + αϕ(z) − γ > 0 (2.3) and

$$\varphi(z) = \frac{f(z)}{z^p}, \qquad (2.2)$$

$$\frac{F(z)}{\alpha - \gamma} \in \mathcal{H}[1, k].$$

By Theorem 1.1

$$\frac{F(z)}{\alpha - \gamma} = \int_\chi \frac{1 + xz}{1 - xz} d\mu(x). \qquad (2.4)$$

Equation (2.4) is equivalent to

$F(z) + \gamma = \beta z^2 \phi''(z) + [\alpha + 2\beta] z\phi'(z) + \alpha\phi(z)$

$$= \gamma + (\alpha - \gamma)\int_\chi \left(1 + \sum_{k=1}^{\infty} 2\, x^k z^k\right) d\mu(x).$$

and (2.4) can also be represented as

$$F(z) + \gamma = [\beta z^2 \varphi'(z)]' - [\alpha z \varphi(z)]' = \int_\chi \left\{\alpha + (\alpha - \gamma) \sum_{k=1}^{\infty} 2x^k z^k\right\} d\mu(x). \qquad (2.5)$$

Integrating both side of (2.5) with respect to z from 0 to z, gives

$$z^2\varphi'(z) + \frac{\alpha}{\beta} z\varphi(z) = \frac{1}{\beta}\int_\chi \left\{\alpha z + (\alpha - \gamma) \sum_{k=1}^{\infty} \frac{2x^k}{k+1} z^{k+1}\right\} d\mu(x). \qquad (2.6)$$

Multiply both sides of (2.6) by $z^{\frac{\alpha - 2\beta}{\beta}}$ and integrating both sides of the result with respect to z from 0 to z, to get

$$\varphi(z) = \int_\chi \left\{1 + (\alpha - \gamma) \sum_{k=1}^{\infty} \frac{2x^k}{(k+1)(\alpha + k\beta)} z^k\right\} d\mu(x). \qquad (2.7)$$

Hence substituting (2.2) in (2.7) yields (2.1). The converse of this deduction process holds and this shows that

$$f \in \mathcal{Q}^{(2)}_{(\alpha,\beta;\gamma)}(f(z), f'(z), f''(z); p).$$

Corollary 2.2 *Let $f \in A(p)$ be at least twice differentiable and $f \in$ $\mathcal{Q}^{(2)}_{(\alpha,\beta;\gamma)}(f(z), f'(z), f''(z); p)$ be as defined in (1.4), then*

$$|a_k| \le (\alpha - \gamma)\left[\frac{2\alpha}{(k+1)(\alpha + k\beta)}\right]. \qquad (2.8)$$

Equality holds for the function f given by

$$f(z) = z^p + (\alpha - \gamma)\sum_{k=1}^{\infty} \frac{2\alpha}{(k+1)(\alpha + k\beta)} z^{k+p-1}. \qquad (2.9)$$

3. The criterion for *p*-convexity

In this section we use some analytic connections among $\frac{f(z)}{[z^p]}$, $\frac{f'(z)}{[z^p]'}$ and $1+\frac{f''(z)}{f'(z)}$ to show the function $f \in C(p)$.

Theorem 3.1 *Let $f \in A(p)$ be at least twice differentiable and $f \in$ $\mathcal{Q}^{(2)}_{(\alpha,\beta;\gamma)}(f(z), f'(z), f''(z); p)$ be as defined in (1.4). A function f' is in $\mathcal{Q}^{(2)}_{(\alpha,\beta;\gamma)}(f(z), f'(z), f''(z); p)$ if and only if f' can be expressed as*

$$f'(z) = pz^{p-1} + 2(\alpha - \gamma)\int_\chi \frac{(k+p)}{(k+1)(\alpha + k\beta)} x^k z^{k+p-1}] d\mu(x), \qquad (3.1)$$

where $\{\mu\}$ is probability measure on χ such that $\chi = \{x: |x| = 1\}$.

Proof Let $f \in A(p)$ with $f \in \mathcal{Q}^{(2)}_{(\alpha,\beta;\gamma)}(f(z), f'(z), f''(z); p)$. We set

$$\phi(z) = \frac{f'(z)}{[z^p]'}, \tag{3.2}$$

then by (1.4)

$$G(z) = z\beta p\phi'(z) + p[\alpha + \beta(1-p)]\phi(z) + \frac{(p\beta - \alpha)(p-1)[f(z)]}{[z^p]} - \gamma > 0 \tag{3.3}$$

and

$$\frac{G(z)}{\alpha - \gamma} \in \mathcal{H}[1, k]. \tag{3.4}$$

Also

$$\beta p\left[z\phi'(z) + \frac{[\alpha + \beta(1-p)]}{\beta}\phi(z)\right] = p[\alpha + \beta(1-p)] + 2(\alpha - \gamma)\int_X \frac{(k+p)[\beta(k+1-p)+\alpha]}{(k+1)(\alpha + k\beta)} x^k z^k d\mu(x) \tag{3.5}$$

Multiplying equation (3.5) by $z^{\frac{\alpha - \beta p}{\beta}}$ and integrating both sides with respect to z from, 0 to z gives,

$$\phi(z) = 1 + 2(\alpha - \gamma)\int_X \frac{(k+p)}{p(k+1)(\alpha + k\beta)} x^k z^k d\mu(x). \tag{3.6}$$

and simply calculations on (3.6) yields (3.1).

If (3.1) holds, reverse calculation shows that $f \in \mathcal{Q}^{(2)}_{(\alpha,\beta;\gamma)}(f(z), f'(z), f''(z); p)$.

Theorem 3.2 *Let* $f \in A(p)$ *be at least twice differentiable and* $f \in Q^{(2)}_{(\alpha,\beta;\gamma)}(f(z), f'(z), f''(z); p)$ *be as defined in (1.4), with* $\alpha \geq 1$, $0 < \beta < 1$, $\alpha > 2\beta p$ *then,* $f \in C(p)$.

Proof Let $f \in A(p)$ with $f \in \mathcal{Q}^{(2)}_{(\alpha,\beta;\gamma)}(f(z), f'(z), f''(z); p)$. We set

$$\vartheta(z) = 1 + \frac{zf''(z)}{f'(z)}, \tag{3.7}$$

then by (1.4)

$$h(z) = \frac{\beta p(p-1)f'(z)(\vartheta(z) - 1)}{z[z^p]''} + \frac{[\alpha + 2\beta(1-p)]p[f(z)]'}{[z^p]'} + \frac{(p\beta - \alpha)(p-1)[f(z)]}{[z^p]} - \gamma > 0,$$

and

$$\frac{h(z)}{\alpha - \gamma} \in \mathcal{H}[1, k]. \tag{3.8}$$

Hence

$$\frac{f'(z)[\beta\vartheta(z) - 2\beta p + \beta + \alpha]}{z^{p-1}} = p[\alpha + \beta(1-p)] + 2(\alpha - \gamma)\int_X \frac{(k+p)[\beta(k+1-p)+\alpha]}{(k-1)(k\beta + \alpha)} x^k z^k d\mu(x). \tag{3.9}$$

Let

$$T(z) = \frac{f'(z)[\beta\vartheta(z) - 2\beta p + \beta + \alpha]}{z^{p-1}}$$

then

$$\mathcal{R}e\left\{z^{p-1}\frac{T(z)}{f'(z)}\right\} = \mathcal{R}e\left\{\frac{[\beta\vartheta(z) - 2\beta p + \beta + \alpha]}{1}\right\}$$

$$\geq \mathcal{R}e\left\{\frac{z^{p-1}}{f'(z)}\frac{f'(z)}{z^{p-1}}\right\},$$

where $\frac{f'(z)}{z^{p-1}}$ is Equation (3.1) divided by z^{p-1}. Hence

$$Re\{\vartheta(z)\} > 0.$$

Hence, Theorem 3.2 is proven.

4. Conclusion

In this article, we calculated the necessary and sufficient condition for

$f \in \mathcal{Q}^{(2)}_{(\alpha,\beta;\gamma)}(f(z), f'(z), f''(z); p)$ and estimated the extreme functions of $f \in \mathcal{Q}^{(2)}_{(\alpha,\beta;\gamma)}(f(z), f'(z), f''(z); p)$. We also showed that if f is in the family $\mathcal{Q}^{(2)}_{(\alpha,\beta;\gamma)}(f(z), f'(z), f''(z); p)$, then $f \in C(p)$.

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