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On structure analysis of finitely generated abelian groups

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Abstract

In this paper we give an overview of the structural analysis of finitely generated abelian groups and modules. We give an overview of recent results on the structure theory of these objects in various situations, in particular in the case of torsionfree groups of infinite rank. We also mention several open problems. We also propose a method to study the structure of finitely generated abelian groups and finitely generated nilpotent groups by using some invariants of their derived series. This method provides a new framework for studying the structure of finitely generated abelian groups and finitely generated nilpotent groups.

Keywords: Groups; Modules; Structure; Finitely generated

1. Introduction

The structure of finitely generated abelian groups has been well understood [18] and [6]. The structure theorem of finitely generated abelian groups states that every finitely generated Abelian group is isomorphic to an internal direct product of cyclic groups [31] and [25]. The classification of finitely generated nilpotent groups in terms of their class semigroup is studied [10] and [27]. Moreover; finitely generated abelian groups and finitely generated nilpotent groups are classified by their type semigroup [3] and [29]. The structure of finitely generated groups has been studied by many authors using different methods [9] and [14]. Some other researchers studied the structure of finitely generated nilpotent groups using the semi group theory and obtained the structure of a finitely generated nilpo tent group using its type semi group [28] and [17]. Some researchers studied the structure of finitely generated group from the point of view of group theory entropy [20] and [4]. The structure of finitely generated nilpotent groups and finitely generated abelian groups was also studied [12] and [33]. The finite direct product of groups is a prototypical example of a group that has finite structure and [30]. In other words; the study of the finite direct product of groups plays a pivotal role in the classification of finite groups [5]; [13] and [16].

Some researcher used the semi-group theory in the description of finitely generated abelian groups and finitely generated nilpotent groups [21] and [2]. The structure of finitely generated nilpotent groups is investigated using homomorphisms of groups [24] and [29]. Some researchers stud ied the structure of finitely generated groups by taking a class of group of the same type [8] and [35]. These groups are called of the same class group type [15] and [26]. The structure of finitely generated groups is also studied using the structure of finitely generated nilpotent groups [23] and [32]. One of the most important problem about finitely generated Abelian group is to classify abelian group of a given type (see; for exam ple; [11] and [19]. There are several algorithms of classifying abelian groups and nilpotent groups of a given type but in many cases; it is computational large [1] and [7].

In this paper we give an overview of the structural analysis of finitely generated abelian groups and modules. We give an overview of recent results on the structure theory of these objects in various situations; in particular in the case of

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torsionfree groups of infinite rank. We also mention several open problems. We define the notion of rank and dimension of modules and groups and give several equivalent definitions of rank and dimension for modules and groups. In the course of the exposition we give a complete answer to the question of the rank of the direct product of two abelian groups. We also discuss a recent idea of rank for finitely generated abelian groups and give some related results. We give the structure of finitely generated torsionfree abelian groups with small rank and also characterize torsionfree nilpotent groups with small rank. We begin with some preliminary remarks on elementary properties of finitely generated abelian groups and modules; introduce the notion of structure theorem for finitely generated abelian groups; and discuss some examples. We also propose a method to study the structure of finitely generated abelian groups and finitely generated nilpotent groups by using some invariants of their derived series. This method provides a new framework for studying the structure of finitely generated abelian groups and finitely generated nilpotent groups.

2. On The Fundamental Theorem and Classification of KNOTS

The fundamental theorem of finitely generated Abelian group is an important theorem in group because it is very useful; effective and con venient for analyzing the structure of a finite abelian group (module). Sometimes we cannot reduce a quotient group without using it; as we have found many examples in the computation of homology groups of Δ−complexes or cellcomplexes in algebraic topology.This theorem has also been used to analyze the symmetry group of composite links L1]L2.

2.1. Theorem 2.1

(Fundamental theorem of finitely generated Abelian group; Poincaré). Let G be a finitely generated abelian group of order n.

$$\text{Then } G \sim= Zn \oplus Zn \oplus Zn \oplus \cdots \oplus Zn;$$

Where;
Zn is a cyclic group of order

$$ni \text{ for } 1 \leq i \leq r \text{ and } 1 \leq n1 \mid n2 \mid \cdots \mid nr.$$

The fundamental theorem of finitely generated abelian groups plays an important role in the group structure analysis of finitely generated abelian groups. The torsionfree rank of the class semigroup of a finitely generated abelian group G is ω(G). Let A be a finitely generated abelian group and H a subgroup of A. Define a function $\phi : H \rightarrow A$ by

$$\phi(h) = ah; \qquad h \in H; a \in A.$$

The function ϕ is a group homomorphism and the kernel of ϕ is H ∩ (H + h0); where h_0 is the neutral element of H. The intersection H ∩(H +h0) is a normal subgroup of H and H/kerϕ ∼= ϕ(H). Therefore

ϕ induces an isomorphism

$$\phi: H/\ker\phi \rightarrow A;$$

And hence

A is isomorphic to the internal direct product of the sub groups.

Let A be a free Abelian group of rank n; and let A have a direct sum decomposition A = K1 ... Km; then each Ki; 1 ≤ i ≤ m; is an Abelian subgroup of A.

In the following part; we will classify a knot using the fundamental theorem of finitely generated abelian groups. We introduce two knots which have different fundamental groups as follows:

K(A(m; k); B(m; k; n).

We define a knot K in S3 by considering C as the connected sum of two torus knots of type (m; k); denoted by T (m; k); and A as the con nected sum of torus knots of type $(m;k)$ and of type (1; n); denoted by T (m; k; n).

The knot K is welldefined since each torus knot is determined up to ambient isotopy. Note that the above definition is not enough to determine the knot K. To give a definition of knot; we should specify each curve of knot. In this paper we consider a torus knot of type (m; k) as a torus knot which is parameterized on the torus T (1; 1) = S1 × S1 by (x; y) ›→ (mx; my).

We assume that the coordinates x and y represent the longitude and the meridian of the torus knot; respectively. The knot type is not changed by different choice of the coordinates of the torus knot; so that we consider a knot of this type as the knot K (A (m; k); B (m; k; n)) for any integer n.

We use the same notation T (m; k) as the torus knot of type (m; k) and as an element of fundamental groupπ 1 (T (1; 1)) = π 1 (S1 × S1) ~= Z ⊕ Z. Hereafter we fix a knot K in S^3 as follows:

$$K(A(m; k); B(m; k; n)) :=(T (m; k) \cup A)](T (m; k) \cup B)$$

$$=T (m; k; n).$$

Hereafter we will use the notation K to denote K(A(m; k); B(m; k; n)) unless we specify the other knots.

Then; in what follows; we consider the knot group and the fundamen tal group of K(A(m; k); B(m; k; n)). First; we consider the knot group

π1(K) of K. We define the group homomorphism ψ : Z ⊕ Z → Z ⊕ Z as follows:

$$\psi(x) =mx$$

$$\psi(y) =ky.$$

We will denote ψ(x) and ψ(y) by a*j* and b*j*; respectively.

Then the knot group π1(K) of K is isomorphic to the group Z ⊕ Z under the following isomorphism:

$$\psi-1 \circ \phi: Z \oplus Z \to Z \oplus Z$$

$$(x; y) \mapsto (aj; bj).$$

In a similar way; we consider the fundamental group π1 (S3 − K). We identify T (1; 1) × [0; 1] ⊂ S3 and then we can define the fundamental group π1 (S3 − K) of S3 − K as π1 (S3 − K) = π1 (T (1; 1)) = Z ⊕ Z.

Then we can define the group homomorphism φ: Z ⊕ Z → Z ⊕ Z as

Follows:

$$\varphi(x) =x + 1$$

$$\varphi(y) =(mx - 1) - (kx + k + 1).$$

We remark that we can easily check these results using the Alexander polynomials.

- Remark 2.2. The fundamental theorem of finitely generated abelian groups can be generalized for finitely generated modules over a princi pal ideal domain.
- Remark 2.3. The fundamental group of a closed surface can be analyzed by the structure theorem. In particular; the fundamental group of a closed surface with trivial fundamental group is a finitely generated abelian group; and the fundamental group of the closed 2dimensional sphere and torus are both finitely generated abelian groups.
- Remark 2.4. Structural analysis of groups or modules can be used to classify knots and links and therefore has applications in DNA and beyond to identify synthetic molecules that can potentially be used in tests for viral infections; such as nucleic acid tests for COVID 19 [36]; [22] and [34].

Compliance with ethical standards

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Data Availability Statement

The author confirms that the data supporting the findings of thisstudy are available within the article or its supplementary materials.

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