Comprehensive review of numerical schemes based on Hermite wavelets

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Abstract

Differential and integral equations are encountered in many applications of science and engineering and many mathematical models have also been formulated in terms of these equations. Due to some shortcomings of the already existing numerical methods, researchers are making efforts to find more efficient alternatives for obtaining solutions of many practical and physical problems giving rise to differential or integral equations. As a result, wavelet methods have found their way for the numerical solution of the resulting different kinds of equations. So this review paper intends to provide the great utility, accuracy and employability of Hermite wavelets to address situations of various areas of applied mathematics, physics, biology, optimal control systems, communication theory, queuing theory, medicine and many other scientific and engineering problems.

Keywords: Hermite wavelets; Collocation points; Operational matrices; Function approximation

1. Introduction

Since differential and integral equations are one of the essential tools for various areas of applied mathematics, there arises an acute requirement to emphasize on the importance of interdisciplinary effort and computational approach so as to advance the study of solving such equations for further scientific research. In the past literature, many research papers have been published to present and establish different numerical methods for solving various types of differential and integral equations. Besides some iterative methods like Homotopy perturbation method (HPM), Adomian decomposition method (ADM) and Variational Iteration Method (VIM), various conventional methods such as Fourier spectral method, Galerkin method, collocation method, finite element method and finite difference method have been mentioned and used to solve linear and nonlinear differential or integral equations. But numerical methods based on Hermite wavelets are observed to provide the high precision numerical solution. Thus wavelets theory is a newly emerging area and a budding tool in mathematical research area and in many different fields of science and engineering. Basically, wavelets are functions that “wave” above and below the x-axis and have varying frequency, limited duration and an average value of zero. These localized or small waves permit the accurate representation of a variety of functions and operators. A very special feature of the wavelets basis functions \( \psi_{ij}(x) \) is that all these functions are considered from a single mother wavelet \( \psi(x) \) which is a small pulse.

An efficient Hermite wavelet based high resolution method is proposed for solving space-time-fractional partial differential equations (STFPDE) in Faheem et al. [1]. A fractional integral operator for Hermite wavelets is utilized and a comparative study of the numerical results with those obtained from other methods validates the authenticity of the proposed method. Kumbinarasaiah [2] solved multidimensional fractional coupled Navier–Stokes equation (NSE) with the help of efficient Hermite wavelets based method along with collocation points. The numerical solutions thus obtained for integer as well as for fractional order NSE are depicted through tables and graphs to validate the efficiency of the proposed technique Hermite wavelet collocation method has been applied to find the numerical solution of
nonlinear ordinary differential equation (ODE) representing the variation of temperature in a permeable moving fin of the rectangular domain in Raghunatha & Kumbinarasaiah [3]. Kumbinarasaiah & Raghunatha [4] proposed Hermite wavelet based method to get solutions of the highly nonlinear Jeffery–Hamel flow problem by utilizing the developed operational matrix of integration. The outcomes indicate that this method is more suitable and correct as compared to the already existing methods. In the same way, Angadi [5] illustrated Hermite wavelet based Galerkin method (HWMG) to be more satisfactory when compared with the exact solution and the solutions obtained by existing methods to solve Singular boundary value problems (SBVPs), which occur recurrently in various branches of applied mathematics and chemical sciences. Kumbinarasaiah & Adel [6] exhibited a wavelet based technique by utilizing Hermite polynomials for obtaining an approximate solution of the nonlinear Rosenau–Hyman equation, one of the important applications in the study of fluid mechanics. The application of the portrayed method to various problems reveals its practicability and accuracy in comparison to other existing methods. Kumar & Ghosh [7] solved the famous biological model Lotka-Volterra equations by using Hermite wavelets method (HWM) and also by spectral collocation method (SCM) with shifted Chebyshev polynomials of first kind. A new numerical method based on Hermite wavelets has been formulated for solving a system of ordinary differential equations of integer and fractional orders by Kumbinarasaiah et al. [8]. Kumbinarasaiah [9] has developed an efficient modus of Hermite wavelets collocation method (HWCM) and fractional derivatives of functions for obtaining solutions of multi-term fractional differential equations (MTFDEs). An improved high resolution method based on Hermite wavelets is described in Faheem et al. [10] for finding solutions of space–time-fractional partial differential equations (STFPDE). Kumbinarasaiah & Mundewadi [11] have designed a computationally attractive method based on Hermite wavelet and operational matrix of integration and applied it to solve first-order linear and nonlinear integro-differential equations with the initial condition. Verma & Tiwari [12] proposed new classes of SBVPs which deal with exothermic reactions. Singular nonlinear BVPs are solved with the help of four different methods based on Hermite Wavelets and Haar wavelets coupled with quasi-linearization approach and Newton approach. The error analysis also justifies the preference of Hermite wavelets method over the Haar wavelets method. In Kumbinarasaiah & Raghunatha [13], two nonlinear heat transfer problems which are of great importance in mechanical engineering, have been solved by implementing Hermite wavelet method (HWM). The differential equations are converted to the corresponding algebraic system of nonlinear equations and the results hence achieved are favorably compared with the outcomes of other methods. Kumar et al. [14] applied Hermite wavelets basis and the operational matrix to solve a fractional order mathematical model describing the spread of COVID-19 disease. The graphical results indicate that susceptible peoples catch the infection through direct contact with infected individuals, and indirectly through the presence of coronavirus in the environment. It has been further concluded through study that there is no better option other than social distancing, isolation, and medical treatment to combat the transmission of this disease. A system of nonlinear differential equations arising in a micropolar rotating nanofluid between two parallel plates has been investigated to find solutions with the help of by Hermite wavelet technique (HWT) by Kumbinarasaiah et al. [15]. Appropriate parameters are taken for the fundamental governing equations to discuss the behavior of velocity, temperature, concentration, and micro rotation and the findings prove the capability of the described method. Shiralashetti & Hanaji [16] suggested Hermite wavelet collocation technique to solve singularly perturbed non-linear Benjamina-Bona-Mohany partial differential equation with two parameters. The equation is transformed into system of algebraic equations which are further solved to obtain the required unknown Hermite wavelet coefficients. The numerical solutions thus secured by substituting these coefficients, are interpreted through figures and tables in order to compare with the outcomes of the already existing methods. Kumbinarasaiah [17] solved multi-term fractional differential equations (MTFDEs) by applying Hermite wavelets collocation method (HWCM) and fractional derivatives of functions.

Khan et al. [18] has discussed two computational methods based on Hermite wavelets and Bernoulli wavelets for the solution of linear, nonlinear, nonlinear singular (Emden–Fowler type) and third-order IVPs and BVPs. Faheem et al. [19] illustrated Hermite wavelet, Legendre wavelet, Chebyshev wavelet and Laguere wavelet based collocation methods for finding solutions of neutral delay differential equations. The discussed method converges fast and the results are compared favourably with those of Runge–Kutta-type methods. After the approval of many theories to be ready for mathematical implementation, Discrete Hermite Wavelet Transform was utilized in image processing by Abdulrahman et al. [20]. Mohammed et al. [21] designed a novel Hermite wavelets (HW) technique to solve nonlinear singular periodic boundary value problems. A proficient solver for the coupled system of fractional differential equations (FDEs) is illustrated by Khader & Babatin [22]. Shiralashetti & Hanaji [23] explained and implemented Hermite wavelet method to find solutions of nonlinear singular initial value problems by using properties of Hermite wavelets and collocation points to convert nonlinear singular initial value problems with initial conditions to systems of nonlinear algebraic equations, which can be solved by using suitable methods. A new method is formulated for solving multidimensional stochastic Itô-Volterra integral equations (MSIVIE) using Hermite wavelets (HW) and a stochastic operational matrix of HW (SOMHW) by Shiralashetti & Lamani [24]. Samadyar et al. [25] have characterized a semi-discretization scheme to find the numerical solution of two dimensional (2D) stochastic time fractional diffusion-wave equation. The described procedure can be divided into two parts - first Crank-Nicolson and linear spline techniques are used to discrete the
given problem in the time direction and then the approximate solution in each time step is acquired with the Hermite wavelet based approach. Kheirabadi et al. [26] established a direct numerical method based on Hermite wavelets and operational matrices of integration, for deriving solutions of optimal control problems. New classes of SBVPs which deal with exothermic reactions are proposed by Verma & Tiwari [27]. The computational reliability of these schemes is verified by implementing on five real life examples. Khashem [28] constructed Hermite wavelets function and operational matrix of integration to find an approximate solution of Bratu’s problem, one of the important applications in applied mathematics. Illustrated examples explicitly reflect the efficiency of the proposed method. Mundewadi & Kumbinarasaiyah [29] designed Hermite wavelet method for solving Abel’s integral equations of first and second kind by reducing them into a system of algebraic equations. Shiralashetti & Kumbinarasaiyah [30] introduced a new generalized operational matrix of integration based on Hermite wavelets to find solutions of nonlinear singular boundary value problems. The results of the numerical investigations corroborate the practicability and efficacy of the illustrated methodology. A new Hermite Wavelet Method (HWM) is presented by Shiralashetti & Kumbinarasaiyah [31] to solve linear and nonlinear singular initial and boundary value problems for different physical conditions. The given differential equations are transformed into system of algebraic equations, which are solved by using Newton’s iterative method through Matlab. The Hermite Wavelet based Galerkin method has been designed by Shiralashetti et al. [32] for finding the numerical solution of one dimensional partial differential equation. Shiralashetti et al. [33] described a promising numerical method based on Hermite wavelets for finding the solution of second order delay differential equations, which play an important role in various fields of mathematical modelling. Mundewadi [34] illustrated Hermite wavelet collocation scheme for the numerical solution of Volterra, Fredholm, mixed Volterra-Fredholm integral equations, integro-differential equations and Abel’s integral equations.

Pirum & Ayaz [35] applied a useful and new Hermite Collocation Method (HCM) for solving higher order fractional differential equations with variable coefficients, which appear in mathematical models of various physical processes. Oruc [41] depicted a Hermite wavelets based numerical procedure for obtaining solutions of two-dimensional hyperbolic telegraph equation by rewriting it as a system of partial differential equations after introducing a new variable. Shiralashetti & Kumbinarasaiyah [37] utilized Hermite wavelets to generate new operational matrix of integration, thus, giving birth to Hermite wavelets operational matrix method (HWOMM) for solving nonlinear singular initial value problems of second order. Numerical solutions derived from some illustrative problems are comparing favourably with the errors and exact solutions. Shiralashetti et al. [38] solved one dimensional elliptic problems with the aid of modified Hermite Wavelet based Galerkin Method, and the test examples prove the reliability of the demonstrated method. An effective and reliable method based on Hermite Wavelets has been proposed by Shiralashetti et al. [39] for solving linear and nonlinear Delay Differential Equations (DDEs). A novel technique is introduced by Shiralashetti & Kumbinarasaiyah [40] to obtain solutions of nonlinear Lane–Emden type equations using Legendre, Hermite and Laguerre wavelets. The method is based on the idea of transforming the given differential equation into a system of linear or nonlinear algebraic equations with unknown coefficients. Oruc [41] described a Hermite Wavelets based computational method for solving two-dimensional Sobolev and regularized long wave equations, which usually appear in the flow of fluids. Illustrative problems are included whose results are compared favourably with already existing methods. Shiralashetti & Mundewadi [42] accomplished a method based on Hermite Wavelets for solving Fredholm Integral Equations of the Second Kind by utilizing the Hermite wavelet matrix. Pirum & Ayaz [43] constructed Hermite Collocation Method (HCM) to find solution for a system of fractional order differential equations with variable coefficients. In order to check the employability and accuracy of this method, some numerical applications are also included. Hermite wavelet collocation method, based on the Hermite polynomials, is introduced by Shiralashetti & Mundewadi [44] for solving Fredholm integral equations of the second kind by reducing them into a system of algebraic equations. Gupta & Ray [45] demonstrated an appropriate and expedient method based on the Hermite wavelet expansion, to solve a coupled system of nonlinear time-fractional Jaulent–Miodek (JM) equations. The presented technique is based on the Hermite wavelet expansion and the operational matrices of fractional derivative and integration derived from Hermite wavelets. Abdulrehman [46] constructed Hermite wavelets functions and their operational matrices of integration, in order to present an algorithm for solving nth order Volterra integro differential equations (VIDE). Saeed & Rehman [47] formulated a new method, which is an amalgam of the Method of Steps and Hermite Wavelet Method, and is implemented to find solutions of fractional delay differential equations.

Ayaaz et al. [48] demonstrated the successful application of Hermite Wavelets Method (HWM) for finding numerical solutions of linear and nonlinear boundary value problems of fifth and sixth order. Usman & Mohyud-Din [49] elucidated the Physicists Hermite Wavelet Method (PHWM), which is the modified version of the Legendre Wavelet Method (LWM). Numerical results derived in the illustrative examples certainly implicate the accuracy and validity of the described scheme. A Hermite Collocation Method was suggested by Nilay et al. [50] for obtaining solutions of high-order linear Fredholm integro-differential equations with variable coefficients. The accuracy of the developed scheme and analysis of error are explicitly featured in the included numerical experiments. A trigonometric Hermite wavelet Galerkin method is presented by Gao & Jiang [51] for achieving solutions of the Fredholm integral equations with weakly
singular kernel. The kernel function of the considered integral equation can be divided into two parts, a weakly singular kernel part for which this approximation part is implemented and a smooth kernel part to be dealt with the developed trigonometric. Maleknejad & Yousefi [52] used the wavelet bases of Hermite cubic splines in order to obtain solutions of the integral equations of the second kind. The results derived by using computational methods authenticate the benefit of the wavelet basis.

2. Hermite Polynomials and their properties

The Hermite polynomials $H_n(x)$ of order $n$, are the solutions of Hermite's differential equation, given by

$$y'' - 2xy' + 2ny = 0,$$

Where $n = 0, 1, 2, 3, .....$

These polynomials $H_n(x)$, are given by the Rodrique's formula

$$H_n(x) = (-1)^ne^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$$

And are defined in the interval $(-\infty, \infty)$. The first few Hermite polynomials are

$H_0(x) = 1, H_1(x) = 2x, H_2(x) = 4x^2 - 2,$

$H_3(x) = 8x^3 - 12x, H_4(x) = 16x^4 - 48x^2 + 12,$

$H_5(x) = 32x^5 - 160x^3 + 120x,$

$H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120, etc$

The Hermite polynomials satisfy the recurrence relations

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$$

And

$$H_n(x) = 2nH_{n-1}(x), \text{ where } n=1, 2, \ldots \ldots .$$

The generating function for Hermite polynomials is defined as

$$e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$$

3. Hermite wavelets [37]

Wavelets form a family of mathematical functions $\psi_{a,b}$ derived from one single function $\psi$ called the mother wavelet, by dilation by 'a' (i.e. by a change of scale $a$) and a translation by 'b' (i.e. by a change of position by $b$):

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}} \psi \left( \frac{x-b}{a} \right), \quad a > 0, \quad b \in R$$

If we fix the parameters $a$ and $b$ to the distinct values given by

$$a = a_0^{-k}, b = nb_0a_0^{-k}, a_0 > 1, b_0 > 0,$$

We obtain the following family of discrete wavelets

$$\psi_{a,b}(x) = |a|^{1/2} \psi(a_0^k x - nb_0), \forall a, b \in R, a \neq 0,$$
Where;
\[ \psi_{k,n} \] form a wavelet basis for \( L^2(R) \).

In particular, when \( a_0 = 2 \) and \( b_0 = 1 \), then \( \psi_{k,n}(x) \) forms an orthonormal basis.

Hermite wavelets are defined as,
\[
\psi_{n,m}(x) = \begin{cases} 
\frac{2^{k+1}}{\sqrt{\pi}} H_m(2^k x - 2n + 1), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}} \\
0, & \text{otherwise}
\end{cases}
\]

Where;
\[ m = 0, 1, \ldots, M - 1, \]

\( H_m(x) \) is Hermite polynomial of degree \( m \).

By considering different values of ‘\( n \)’ and ‘\( m \)’, we get the wavelet basis functions. With the help of properties of Hermite wavelets, the differential equations are converted into system of algebraic equations and these resulting equations can afterwards be solved efficiently to obtain better results.

4. Conclusion

From the above review, we conclude that Hermite Wavelets Collocation Method is a powerful and promising tool for investigating and securing solutions of the frequently occurring ordinary and partial differential equations, integral and integro-differential equations. The implementation of Hermite wavelets in solving such equations is simpler and gives better accuracy. But this area is comparatively less explored and so further research and development in this direction will help in proving the employability of this method and contribute in advancement and growth of applied mathematics and many other disciplines.

Compliance with ethical standards

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We claim that there is no conflict of interest between the two authors.

References


