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Certain q-starlike and q-convex functions with respect to conjugate points

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Abstract

Certain subclass of q−starlike and q−convex was introduced . The third and fourth coefficient were calculated, with the aid of subordination theory.

Keywords: q−Derivative operator; Extremal function; Analytic functions; q−Starlike; q−Convex

AMS Subject Classification: 30C45

1. Introduction

Let P denote the class of analytic functions p normalized by

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \tag{1.1}$$

such that Re$\{p(z)\} > 0, z \in$ U.

Let A_k denote the class of functions f normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad , \quad (1.2)$$

which are analytic in the open unit disc U(1) = U where,

U(r) = $\{z : |z| < r\}$.

We state the q-derivative operator $D_{z,q}$ defined by [22] (see also [23]) as

$$\begin{cases} \mathcal{D}_{z,q} f(z) = \frac{f(z)-f(qz)}{z(1-q)}, & q \in (0,1), \ z \neq 0 \\ \mathcal{D}_{z,q} f(z) \mid_{z=0} = f'(0). \end{cases} ,$$

Also, the operator $(D_{z,q}f)(z)$ can also be defined by convolution as follows:

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$$(\mathcal{D}_{z,q}f)(z) = p(z) * \frac{1}{1-[2]_q z + qz^2}, \quad p \in \mathcal{P}, \quad [j]_q = \sum_{\ell=1}^{j} q^{\ell-1}, \ \ell \in \mathbb{N}.$$

We say that the function $\tau : U \to C$ is subordinate to the $\sigma : U \to C$, represented as $\tau \prec \sigma$ or $\tau(z) \prec \sigma(z)$ if there exists a function $\nu : U \to U$, analytic in U such that $\nu(0) = 0$, $|\nu(z)| < 1$, and

$$\tau(z) = \sigma(\nu(z)), \quad z \in U.$$

Motivation and Definition

Aldweby and Darus [7] defined the classes $\mathcal{PS}_q^*(\varphi)$ and $PC_q(\phi)$ by

$$\frac{z\left(\mathcal{D}_{z,q}f\right)(z)}{f(z)} \prec \varphi(z), \quad \varphi \in \mathcal{P}, \ z \in \mathcal{U} \tag{2.1}$$

and

$$1 + \frac{zq\left(\mathcal{D}^2_{z,q}f\right)(z)}{\left(\mathcal{D}_{z,q}f\right)(z)} \prec \varphi(z), \quad \varphi \in \mathcal{P}, \ z \in \mathcal{U} \tag{2.2}$$

respectively. They obtained the second and third coefficient of functions of these classes. Meanwhile, Ramachandran et. al. [34], introduced the classes of q–starlike and q–convex functions with respect to symmetric points, denoted by $\mathcal{PS}_q^*(\varphi)$ and $PC_{q,s}(\phi)$ and defined by

$$\frac{2z\left(\mathcal{D}_{z,q}f\right)(z)}{f(z) - f(-z)} \prec \varphi(z)), \quad \varphi \subset \mathcal{P} \tag{2.3}$$

and

$$\frac{2\left[\left(\mathcal{D}_{z,q}f\right)(z) + zq\left(\mathcal{D}^2_{z,q}f\right)(z)\right]}{\left(\mathcal{D}_{z,q}f\right)(z) + \left(\mathcal{D}_{z,q}f\right)(-z)} \prec \varphi(z), \quad \varphi \in \mathcal{P} \tag{2.4}$$

respectively.

Definition 2.1 *Suppose* $\alpha \in [0,1]$ *and* $\beta \in (0,1)$. *Let* $PS(\alpha,\beta)$ *denote the class of functions* $f \in A_k$ *satisfying the following inequality:*

$$\left|\frac{z(\mathcal{D}_{z,q}f)(z)}{f(z)} - 1\right| < \beta \left|\frac{\alpha z(\mathcal{D}_{z,q}f)(z)}{f(z)} + 1\right|, \quad z \in \mathcal{U}. \tag{2.5}$$

From [37], equation (2.5) can be rewritten as:

$$\frac{z(\mathcal{D}_{z,q}f)(z)}{f(z)} \prec \frac{1+\beta z}{1-\alpha\beta z}, \quad z \in \mathcal{U}. \tag{2.6}$$

If $\alpha = \beta = 1$ then the class $PC(\alpha,\beta)$ reduces to the class $PC(\phi(z))$ studied by [7].

Also if $q \to 1$ the class $PC(\alpha,\beta)$ reduces to the class $S(\alpha,\beta)$ studied by [35].

Definition 2.2 *Suppose* $\alpha \in [0,1]$ *and* $\beta \in (0,1)$. *Let* $PC(\alpha,\beta)$ *denote the class of functions* $f \in A_k$ *satisfying the following inequality:*

$$\left|\frac{\mathcal{D}_{z,q}\left[z(\mathcal{D}_{z,q}f)(z)\right]}{\left(\mathcal{D}_{z,q}f\right)(z)} - 1\right| < \beta \left|\frac{\alpha\mathcal{D}_{z,q}\left[z(\mathcal{D}_{z,q}f)(z)\right]}{\left(\mathcal{D}_{z,q}f\right)(z)} + 1\right|, \quad z \in \mathcal{U}. \tag{2.7}$$

$$\frac{\mathcal{D}_{z,q}\left[z(\mathcal{D}_{z,q}f)(z)\right]}{(\mathcal{D}_{z,q}f)(z)} \prec \frac{1+\beta z}{1-\alpha\beta z}, \quad z \in \mathcal{U}. \qquad (2.8)$$

If $\alpha = \beta = 1$ then the class PC(α,β) reduces to the class PC($\phi(z)$) studied by [7].

Also if $q \to 1$ the class PC(α,β) reduces to the class K(α,β) studied by [35].

Definition 2.3 *Suppose* $\alpha \in [0,1]$ *and* $\beta \in (0,1)$. *Let* $\mathcal{PS}_s^*(\alpha,\beta)$ *denote the class of functions* $f \in A_k$ *satisfying the following inequality:*

$$\left|\frac{2z(\mathcal{D}_{z,q}f)(z)}{f(z)-f(-z)} - 1\right| < \beta\left|\frac{2\alpha z(\mathcal{D}_{z,q}f)(z)}{f(z)-f(-z)} + 1\right|, \quad z \in \mathcal{U} \qquad (2.9)$$

(2.10)

$$\frac{2z(\mathcal{D}_{z,q}f)(z)}{f(z)-f(-z)} \prec \frac{1+\beta z}{1-\alpha\beta z}, \quad z \in \mathcal{U}.$$

Definition 2.4 *Suppose* $\alpha \in [0,1]$ *and* $\beta \in (0,1)$. *Let* PC(α,β) *denote the class of functions* $f \in A_k$ *satisfying the following inequality:*

$$\left|\frac{2\mathcal{D}_{z,q}\left[z(\mathcal{D}_{z,q}f)(z)\right]}{\mathcal{D}_{z,q}\left[f(z)-f(-z)\right]} - 1\right| < \beta\left|\frac{2\alpha\mathcal{D}_{z,q}\left[z(\mathcal{D}_{z,q}f)(z)\right]}{\mathcal{D}_{z,q}\left[f(z)-f(-z)\right]} + 1\right|, \quad z \in \mathcal{U} \qquad (2.11)$$

$$\frac{2\mathcal{D}_{z,q}\left[z(\mathcal{D}_{z,q}f)(z)\right]}{\mathcal{D}_{z,q}\left[f(z)-f(-z)\right]} \prec \frac{1-\beta z}{1-\alpha\beta z}, \quad z \in \mathcal{U}. \qquad (2.12)$$

Lemma 2.1 *Let* $p \in P$ *then the following sharp estimate holds:*

(2.13)

$|p_k| \leq 2, \; k \in N.$

Lemma 2.2 *Let* $p \in P$. *Then*

$$2p_2 = p_1^2 + x\left(4 - p_1^2\right),$$

for some x with $|x| \leq 1$ *and*

$$4p_3 = p_1^3 + 2p_1\left(4-p_1^2\right)x - p_1\left(4-p_1^2\right)x^2 + 2\left(4-p_1^2\right)\left(1-|x|^2\right)z \qquad (2.14)$$

for some z with $|z| \leq 1$.

Necessary and Sufficient Conditions

Theorem 3.1 *Let* $f \in$ PS(α,β), $\beta \in (0,1]$, $\alpha \in [0,1]$, *then*

$$\left|a_2a_4 - a_3^2\right| = \frac{\beta^2(1+\alpha)^2}{q^2\,[2]_q^2}. \qquad (3.1)$$

The result is sharp, with the extremal function

$$f_1(z) = \begin{cases} \frac{z}{(1-\alpha\beta z^2)^{\frac{1+\alpha}{[2]_q}}}, & \alpha \in (0, 1]; \\ z\exp(\frac{\beta z^2}{[2]_q}), & \alpha = 0. \end{cases} \qquad (3.2)$$

Proof. Let $f \in PS(\alpha,\beta)$, then by (2.6) there exists a Schwarz function ν, which is analytic in U such that $\nu(0) = 0$, $|\nu(z)| < 1$ and

$$\frac{z(\mathcal{D}_{z,q}f)(z)}{f(z)} = \vartheta\left(\nu(z)\right), \qquad (3.3)$$

where

$$\vartheta(z) = \frac{1+\beta z}{1-\alpha\beta z} = 1 + \beta(1+\alpha)z + \alpha\beta^2(1+\alpha)z^2 + \alpha^2\beta^3(1+\alpha)z^3 + \cdots.$$

Let $p \in P$ be written as follows:

$$p(z) = \frac{1+\nu(z)}{1-\nu(z)}. \qquad (3.4)$$

Then by (1.1)

$$\nu(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \left(p_3 + \frac{p_1^3}{4} - p_1p_2\right)z^3 + \cdots\right]. \qquad (3.5)$$

And

$$\frac{z(\mathcal{D}_{z,q}f)(z)}{f(z)} = \vartheta\left(\frac{p(z)-1}{p(z)+1}\right)$$

$$= 1 + \frac{\beta}{2}(1+\alpha)p_1 z + \left[\frac{\beta}{2}(1+\alpha)(p_2 - \frac{p_1^2}{2}) + \frac{\alpha\beta^2}{4}(1+\alpha)p_1^2\right]z^2$$

$$+ \left[\frac{\beta(1+\alpha)}{2}\left(\frac{p_1^3}{4} + p_3 - p_1p_2\right) + \frac{\alpha\beta^2(1+\alpha)}{2}(p_2 - \frac{p_1^2}{2})p_1 + \frac{\alpha^2\beta^3(1+\alpha)}{8}p_1^3\right]z^3 \cdots. \qquad (3.6)$$

Also,

$$\frac{z(\mathcal{D}_{z,q}f)(z)}{f(z)} = 1 + qza_2 + qz^2\left([2]_q a_3 - a_2^2\right)$$

$$+ qz^3\left([3]_q a_4 - (1+[2]_q)a_2a_3 + a_2^3\right). \qquad (3.7)$$

Comparing the coefficients of z, z^2 and z^3 in (3.6) and (3.7), we get

$$a_2 = \frac{\beta(1-\alpha)p_1}{2q},$$

$$a_3 = \frac{\beta(1+\alpha)}{4q^2[2]_q}\left[2qp_2 + \left(\alpha\beta[2]_q + \beta - q\right)p_1^2\right]$$

$$a_4 = \frac{\beta\left(1+\alpha\right)}{8\ q^3\ [3]_q\ [2]_q} \times$$

$$\left\{\left[\alpha^2\beta^2[3]_q[2]_q + \alpha\beta^2\left([3]_q + [2]_q\right) + \beta^2 + q^2[2]_q - \alpha\beta q\left(2[3]_q + q\right) + \beta q\left(1+[2]_q\right)\right]p_1^3 + \left[2\alpha\beta q\left(q + 2[3]_q\right) + 2\beta q\left(1+[2]_q\right) - 4q^2[2]_q\right]p_1p_2 + \left[4q^2[2]_q\right]p_3\right\}$$

Thus,

$$a_2a_4 - a_3^2 = -\frac{\beta^2\left(1+\alpha\right)^2}{16\ q^2\ [3]_q\ [2]_q^2} \times$$

$$\left\{\left(\alpha\beta^2[2]_q + \alpha\beta[2]_q + \beta + \beta^2q[2]_q - q\beta + q^2\right)p_1^4\right.$$

$$\left.\left(\beta q + 2q - \beta - \alpha\beta[2]_q\right)2p_1^2p_2 + \left(4[3]_q\right)p_2^2 - \left([2]_q\right)4p_1p_3\right\}.$$

Hence,

$$\left|a_2a_4 - a_3^2\right| = \frac{\beta^2\left(1+\alpha\right)^2}{16\ q^2\ [3]_q\ [2]_q^2} \times$$

$$\left|\left(\alpha\beta^2[2]_q + \beta^2q[2]_q + 2q[2]_q\right)p_1^4 - \left(\alpha\beta[2]_q + \beta(1-q) - 2q[2]_q\right)\left(4-p_1^2\right)p_1^2x\right.$$

$$\left. + \left(4[3]_q - q^2p_1^2\right)\left(4-p_1^2\right)x^2 - \left(4-p_1^2\right)\left(1-|x|^2\right)2[2]_qp_1z\right|$$

$$\left|a_2a_4 - a_3^2\right| \le \frac{\beta^2\left(1+\alpha\right)^2}{16\ q^2\ [3]_q\ [2]_q^2}.$$

$$\left(\alpha\beta^2[2]_q + \beta^2q[2]_q - 2q[2]_q\right)p^4 + \left(\alpha\beta[2]_q + \beta(1-q) - 2q[2]_q\right)\left(4-p^2\right)p^2|x|$$

$$+ \left(4[3]_q - q^2p^2\right)\left(4-p^2\right)|x|^2 + \left(4-p^2\right)\left(1-|x|^2\right)2[2]_qp$$

$$(4-p^2)2[2]_qp + \left(\alpha\beta^2[2]_q + \beta^2q[2]_q + 2q[2]_q\right)p^4 + \left(\alpha\beta[2]_q + \beta(1-q) - 2q[2]_q\right)\left(4-p^2\right)p^2|x|$$

$$+ (2-p)\left\{2[3]_q + q^2p\right\}\left(4-p^2\right)|x|^2$$

$= \mathcal{F}(p,\psi)$,

where $\psi = |x| \le 1$.

$$\frac{\partial\left(F(p,\psi)\right)}{\partial\psi} = \frac{\beta^2\left(1+\alpha\right)^2}{16\ q^2\ [3]_q\ [2]_q^2}\left[\left(\alpha\beta[2]_q + \beta(1-q) - 2q[2]_q\right)\left(4-p^2\right)p^2\right.$$

$$\left. + 2(2-p)\left\{2[3]_q + q^2p\right\}\left(4-p^2\right)|x|\right] \qquad (3.8)$$

$$\max_{\psi\in[0,1]} F(p,\psi) = F(p,1) = \varphi(p)$$

$$F(p,1) = \varphi(p) = \frac{\beta^2(1+\alpha)^2}{16\, q^2\, [3]_q\, [2]_q^2}$$

$$16[3]_q + \left(q\left[q+4[2]_q\right] + \alpha\beta^2[2]_q + \beta^2 q[2]_q + \beta q - \beta - \alpha\beta[2]_q\right)p^4$$

$$\left(\alpha\beta - 2p^2q\right)4[2]_q p^2$$

$$\varphi'(p) = \frac{\beta^2(1+\alpha)^2}{4\, q^2\, [3]_q\, [2]_q^2}$$

$$\left(q\left[q+4[2]_q\right] + \alpha\beta^2[2]_q + \beta^2 q[2]_q + \beta q - \beta - \alpha\beta[2]_q\right)p^3$$

$$\left(\alpha\beta - 2p^2q\right)2[2]_q p$$

$$\left|a_2a_4 - a_3^2\right| = \frac{\beta^2(1+\alpha)^2}{q^2\, [2]_q^2}$$

equality holds for the function

Solving, we have

$$f_1(z) = \begin{cases} \frac{z}{(1-\alpha\beta z^2)^{\frac{1+\alpha}{[2]_q}}}, & \alpha \in (0, 1]; \\ z\exp\left(\frac{\beta z^2}{[2]_q}\right), & \alpha = 0. \end{cases}$$

$$\frac{z\left(D_{z,q}f_1(z)\right)}{f_1(z)} = \frac{f_1(z)}{f_1(zq)}\left(\frac{1+\beta z}{1-\alpha\beta z}\right) \quad (3.9)$$

Theorem 3.2 *Let* $\beta \in (0,1]$, $\alpha \in [0,1]$.

$$\left|a_2a_4 - a_3^2\right| = \frac{\beta^2(1+\alpha)^2}{q^2\, [2]_q^2}. \quad (3.10)$$

The result is sharp, with the extremal function

$$f_1(z) = \begin{cases} \frac{z}{(1-\alpha\beta z^2)^{\frac{1+\alpha}{[2]_q}}}, & \alpha \in (0, 1]; \\ z\exp\left(\frac{\beta z^2}{[2]_q}\right), & \alpha = 0. \end{cases} \quad (3.11)$$

Proof.

Let $f \in PS(\alpha,\beta)$, then by (2.6) there exists a Schwarz function ν, which is analytic in U such that $\nu(0) = 0$, $|\nu(z)| < 1$ and

$$\frac{2z(\mathcal{D}_{z,q}f)(z)}{f(z)-f(-z)} = \vartheta(\nu(z)), \quad (3.12)$$

where

$$\vartheta(z) = \frac{1+\beta z}{1-\alpha\beta z} = 1 + \beta(1+\alpha)z - \alpha\beta^2(1+\alpha)z^2 + \alpha^2\beta^3(1+\alpha)z^3 + \cdots.$$

Let $p \in$ P be written as follows:

$$p(z) = \frac{1+\nu(z)}{1-\nu(z)}. \qquad (3.13)$$

Then by (1.1)

$$\nu(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}\left[p_1 z + \left(p_2 - \frac{p_1^2}{2}\right)z^2 + \left(p_3 + \frac{p_1^3}{4} - p_1 p_2\right)z^3 + \cdots\right]. \qquad (3.14)$$

And

$$\frac{2z(\mathcal{D}_{z,q}f)(z)}{f(z)-f(-z)} = \vartheta\left(\frac{p(z)-1}{p(z)+1}\right)$$

$$= 1 + \frac{\beta}{2}(1+\alpha)p_1 z + \left[\frac{\beta}{2}(1+\alpha)(p_2 - \frac{p_1^2}{2}) + \frac{\alpha\beta^2}{4}(1+\alpha)p_1^2\right]z^2$$

$$+ \left[\frac{\beta(1+\alpha)}{2}\left(\frac{p_1^3}{4} + p_3 - p_1 p_2\right) + \frac{\alpha\beta^2(1+\alpha)}{2}(p_2 - \frac{p_1^2}{2})p_1 + \frac{\alpha^2\beta^3(1+\alpha)}{8}p_1^3\right]z^3 \cdots . \qquad (3.15)$$

Also,

$$\frac{2z(\mathcal{D}_{z,q}f)(z)}{f(z)-f(-z)} = 1 + [2]_q z a_2 + q[2]_q z^2 a_3 + [2]_q z^3\left\{(1+q^2)a_4 - a_2 a_3\right\}\cdots$$

Comparing the coefficients of z, z^2 and z^3

$$a_2 = \frac{\beta(1+\alpha)p_1}{2[2]_q}$$

$$a_3 = \frac{\beta(1+\alpha)}{4q[2]_q}\left\{2p_2 + (\alpha\beta - 1)p_1^2\right\}$$

$$a_4 = \frac{\beta(1+\alpha)}{8q[4]_q[2]_q}\left\{\begin{array}{c} [\ \alpha^2\beta^2[3]_q + \alpha\beta^2 - \alpha\beta\left([3]_q + q[2]_q\right) - \beta + q[2]_q\ ]\ p_1^3 \\ \\ +[\ \alpha\beta\left([3]_q + q[2]_q\right) + \beta - 2q[2]_q\ ]\ 2p_1p_2 + [\ 4q[2]_q\ ]\ p_3 \end{array}\right\}$$

$$a_4 a_2 - a_3^2 = -\frac{\beta^2(1+\alpha)^2}{16q^2[4]_q[2]_q^2}$$

$$\left\{\begin{array}{c} [\ \alpha^2\beta^2 + \beta q + [2]_q - \alpha\beta^2 q - \alpha\beta\left([2]_q + 1\right)\ ]\ p_1^4 + 4p_2^2[4]_q \\ \\ +\left(\alpha\beta(1+[2]_q) - 2[2]_q - \beta q\right)2p_1^2p_2 - [\ 4q^2[2]_q\ ]\ p_1p_3 \end{array}\right\},$$

$$\left|a_4 a_2 - a_3^2\right| = \frac{\beta^2(1+\alpha)^2}{16q^2[4]_q[2]_q^2}$$

$$|[\alpha^2\beta^2 + \beta q + [2]_q - \alpha\beta^2 q - \alpha\beta\left([2]_q + 1\right)]\ p_1^4 + 4p_2^2[4]_q$$

$$\left|a_4a_2 - a_3^2\right| = \frac{\beta^2(1+\alpha)^2}{16q^2[4]_q[2]_q^2}$$

$$[\ \alpha^2\beta^2 - \alpha\beta^2 q\ ]\ p_1^4 + (\alpha\beta(1+[2]_q) - \beta q)\, x\,(4-p_1^2)\, p_1^2$$
$$+ (\alpha\beta(1+[2]_q) \quad 2[2]_q \quad \beta q)\, 2p_1^2 p_2 \quad [\ 4q^2[2]_q\ ]\ p_1 p_3| \qquad (3.16)$$

$$\left| + (4-p_1^2)\,(4\{[3]_q - q\})\,[2]_q x^2 - 2q^2[2]_q p_1\,(4-p_1^2)\,(1-|x|^2)\, z \right|.$$

$$\begin{aligned}|a_4a_2 - a_3^2| \le\ & \frac{\beta^2(1+\alpha)^2}{16q^2[4]_q[2]_q^2} \\ & \times [\ \alpha^2\beta^2 - \alpha\beta^2 q\ ]\ p^4 + (\alpha\beta(1+[2]_q) - \beta q)\,|x|\,(4-p^2)\,p^2 \\ & + (4-p^2)\,(4\{[3]_q - q\} - p^2)\,[2]_q|x|^2 + 2q^2[2]_q p\,(4-p^2)\,(1-|x|^2)\end{aligned} \qquad (3.17)$$

$$= \frac{\beta^2(1+\alpha)^2}{16q^2[4]_q[2]_q^2}\ [\ \alpha^2\beta^2 - \alpha\beta^2 q\ ]\ p^4$$
$$+ 2q^2[2]_q p\,(4-p^2) + (\alpha\beta(1+[2]_q) - \beta q)\,(4-p^2)\,p^2|x|$$
$$(3.18) \qquad + [2]_q\,(4-p^2)\,(4\{[3]_q - q\} - p(p+2q^2))\,|x|^2$$
$$= F(p,\psi) \qquad (say), \qquad \psi = |x| \le 1.$$

$$\frac{\partial F(p,\psi)}{\partial \psi} = \frac{\beta^2(1+\alpha)^2}{16q^2[4]_q[2]_q^2}\,(\alpha\beta(1+[2]_q) - \beta q)\,(4-p^2)\,p^2 \qquad (3.19)$$
$$+ 2[2]_q\,(4-p^2)\,(4\{[3]_q - q\} - p(p+2q^2))\,|x| \qquad (3.20)$$

$$\max_{\psi[0,1)} F(p,\psi) = F(p,1) = \varphi(p), \quad (say).$$

$$\begin{aligned}\varphi(p) = F(p,1) =\ & \frac{\beta^2(1+\alpha)^2}{16q^2[4]_q[2]_q^2} \\ & \times \left(\alpha^2\beta^2 + [2]_q + \beta q - \alpha\beta^2 q - \alpha\beta\,\{1+[2]_q\}\right) p^4 \\ & + (\alpha\beta\{1+[2]_q\} - \beta q - \{[4]_q + [2]_q\})\,4p^2 + 16[4]_q\end{aligned} \qquad (3.21)$$

$$\begin{aligned}\varphi'(p) =\ & \frac{\beta^2(1+\alpha)^2}{4q^2[4]_q[2]_q^2}\,p \\ & \times \left(\alpha^2\beta^2 + [2]_q + \beta q - \alpha\beta^2 q - \alpha\beta\,\{1+[2]_q\}\right) p^2 \\ & + (\alpha\beta\{1+[2]_q\} - \beta q - \{[4]_q - [2]_q\})\,2\end{aligned}$$

$$\left|a_2a_4 \quad a_3^2\right| \le \frac{\beta^2(1-\alpha)^2}{q^2[2]_q^2} \qquad (3.22)$$

Equality holds for the function

(3.23)
$$f_4(t) = \begin{cases} \int_0^t \frac{1}{(1-\alpha\beta\vartheta^2)^{\frac{1+\alpha}{[2]_q q^\alpha}}} \left(\frac{1+\alpha\beta\vartheta^2}{1-\alpha\beta\vartheta^2}\right) d_q\vartheta; & 0 < \alpha \le 1 \\ \int_0^t \exp[\frac{\beta\vartheta^2}{[2]_q}]\,(1+\beta\vartheta^2)\, d_q\vartheta : & \end{cases}$$

$\alpha = 0$.

Solving with $0 < \alpha \le 1$ we have

$$f_4(t) - f_4(-t) = \int_0^t \frac{2}{(1-\alpha\beta\vartheta^2)^{\frac{1+\alpha}{[2]_q}}} \left(\frac{1+\beta\vartheta^2}{1-\alpha\beta\vartheta^2}\right) d_q\vartheta$$

$$= \frac{2t}{(1-\alpha\beta t^2)^{\frac{1+\alpha}{[2]_q}}}. \qquad (3.24)$$

and with $\alpha = 0$ we have

$$f_4(t) - f_4(-t) = 2t\exp[\frac{\beta t^2}{[2]_q}]. \qquad (3.25)$$

$$\frac{2zf_4'(z)}{f_4(z)-f_4(-z)} = \frac{1+\beta z^2}{1-\alpha\beta z^2} \prec \frac{1+\beta z}{1-\alpha\beta z} \qquad (3.26)$$

2. Conclusion

Certain subclass of $q-$starlike and $q-$convex was introduced . The third and fourth coefficient were calculated, with the aid of subordination theory.

Compliance with ethical standards

Disclosure of conflict of interest

The author disclosed there is no conflict of interest.

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