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Some analytic properties of Janowski q-class functions

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Abstract

The q-derivative operator was stated as a convolution of two analytic functions. The necessary and sufficient conditions for a Janowski's harmonic q-starlike functions were studied. Also, some subordination properties of Janowski's analytic q-starlike functions were studied. This article ends with a few open questions.

Keywords: q–Derivative operator; q–Integral operator; Janowski harmonic functions; Janowski analytic functions**.**

AMS Subject Classification: 30C45

1. Introduction

Let A_k denote the class of functions f normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k ,$$

which are analytic in the open unit disc U(1) = U where,

$U(r) = \{z : |z| < r\}$.

The applications of q-derivative operator $D_{z,q}$ defined by [3] (see also [4]) as

$$\begin{cases} \mathcal{D}_{z,q} f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, & q \in (0,1), \; z \neq 0 \\ \mathcal{D}_{z,q} f(z) \mid_{z=0} = f'(0), \end{cases} ,$$

(where $[k]_q = 1+q+\cdots+q^{k-1}$) to the so called q-analysis in Geometric Function Theory of Complex Analysis dates back to late 1980s. It started with the generalization of the class, $S^?$ of starlike functions in U satisfying

$$f'(0) - 1 = f(0) = 0, \quad \mathcal{R}e\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \mathcal{U}$$

1 .

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The generalized class of starlike function called the class of q-starlike functions of f denoted by, was defined by Ismail et al [7] $\mathcal{PS}_q^*$ as the following:

Definition 1.1 *A function $f \in A_k$ is said to belong to the class $\mathcal{PS}_q^*$ if*

$$\left|\frac{z\mathcal{D}_{z,q}f(z)}{f(z)} - \frac{1}{1-q}\right| < \frac{1}{1-q} \quad (1.1)$$

$q \in (0,1)$; $z \in$ U.

Meanwhile, in 1869, Thomae [1] introduced the q-integral operator

$$I_{z,q}^{\alpha,\beta} f(z) = \int_0^1 f(\zeta)\, d_q\zeta = (1-q)\sum_{k=0}^{\infty} q^k\, f(q^k)$$

provided the q-series converges. Jackson [2] also defined the general q-integral operator

$$\mathcal{I}_{z,q}^{\alpha,\beta} f(z) = \int_\alpha^\beta f(\zeta)d_q\zeta := \int_0^\beta f(\zeta)d_q\zeta - \int_0^\alpha f(\zeta)d_q\zeta = z(1-q)\sum_{k=0}^{\infty} q^k f(zq^k)$$

provided the q-series converges. Recently, Agrawal and Sahoo [8] defined and studied the class of functions, $\mathcal{PS}_q^*(\alpha)$ of q-starlike of order alpha. They established some important results which includes Lemma 1.1, stated as:

Lemma 1.1 *Let $f \in A_k$ and $q \in (0,1)$. Then*

$$\mathcal{I}_{z,q}^{0,z}\frac{(\mathcal{D}_{z,q}f)(z)}{f(z)} = \frac{(q-1)}{\ln q} Log f(z) \quad . \quad (1.2)$$

We say that the function τ : U → C is subordinate to the σ : U → C, represented as $\tau \prec \sigma$ or $\tau(z) \prec \sigma(z)$ if there exists the complex-valued function ν : U → U, with $\nu(0) = 0$, such that

$\tau(z) = \sigma(\nu(z))$, $z \in$ U.

Previously Janowski, in 1973 introduced the class of functions $S^*(E,A)$, for arbitrary fixed numbers, E ∈ (−1,1] and A ∈ [−1,1) as follows:

Definition 1.2 *[23]. Let $f \in A_k$, and ν be analytic in* U *with $\nu(0) = 0$,* $|\nu(z)| < 1$. *(E,A)|
if and only if

$$\frac{zf'(z)}{f(z)} = \mathcal{P}(\nu(z)),$$

for some class of functions P *such that*

$P(\nu(z)) = (1 + E\nu(z))(1 + A\nu(z))^{-1}$, $z \in$ U.
Janowski determined among other results the bounds for

$$\mathcal{R}e\left\{p(z) + \frac{zp'(z)}{p(z)}\right\} > 0, \quad \mathcal{R}e\left\{\frac{zp'(z)}{p(z)}\right\} > 0, \quad p \in \mathcal{P}(\mathcal{E},\mathcal{A}).$$

He also determined the bounds for $|f(z)|$ and $|f^0(z)|$ of the function $f \in S^*(E,A)$. Many authors like [18] - [28] to mention but a few had studied some properties of functions in the family $S^*(E,A)$.

Motivated by some applications of q-calculus to the Geometric Function Theory of Complex Analysis introduced and studied by [7], and by many authors like [5] -[18] to mention but a few, in this article we extend the study of q-calculus to some subordination properties of the Janowski's class of harmonic and analytic functions.

The main aim of this article is to define and study the followings:

Define

the q-derivative operator $D_{z,q}$on $f \in A_k$ using the convolution product of two analytic functions,

the Janowski's class of harmonic q-starlike functions $\mathcal{PS}^*_{q,\mathcal{H}}(\mathcal{E},\mathcal{A})$

and the Janowski's class of analytic q-starlike functions $\mathcal{PS}^*_{q,\mathcal{P}}(\mathcal{E},\mathcal{A})$.

Study

the necessary and sufficient conditions for the function $f \in S_H$ to be in the class $\mathcal{PS}^*_{q,\mathcal{H}}(\mathcal{E},\mathcal{A})$

the sufficient condition for the function $f \in H_0$ to be in the class

$\mathcal{PS}^*_{q,\mathcal{H}}(\mathcal{E},\mathcal{A})$

the necessary and sufficient conditions for the function $f \in PJ^0$ to be in the the class $\mathcal{PS}^*_{q,\mathcal{J}}(\mathcal{E},\mathcal{A})$

and calculate some subordinate properties of the Janowski analytic q-starlike functions.

Preliminaries Concepts of the q-Class of Janowski Functions

Firstly, we let H denote the class of harmonic functions in the unit disc U and by H_0 we denote the class of normalized by $f(0) = f'_z(0) = f'_{\bar{z}}(0) - 1 = 0$. The function $f \in H_0$ can be written as

$$f = h + \bar{g}, \quad (2.1)$$

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where both h and g are analytic . Also h and g can be expressed as

$h(z) = z + \sum_{k=2}^{\infty} a_k \ z^k$ and $g(z) = \ b_k z^k$, $|b| < 1$,

that means

$$f(z) = \sum_{k=1}^{\infty} \left(a_k z^k + \overline{b_k z^k} \right), \quad a_1 = 1, \ |b_1| < 1, \ z \in \mathcal{U}, \ f \in \mathcal{H}_0 \quad . \ (2.2)$$

The function h is called the analytic part while g is the co-analytic part of f. By S_H we denote the class of functions $f \in H_0$ which are univalent and sense-preserving in U. We note that the class of functions $f \in A_k$ is the same as the class of functions $f \in H$ for which the co-analytic part vanish. Furthermore, a necessary and sufficient condition for the function $f \in H_0$ to be locally univalent and sense preserving in U is that $\left|\frac{h'(z)}{g'(z)}\right| > 1$.

Secondly, the q-derivative operator $D_{z,q}$on the function $f \in A_k$ can be presented as the convolution of two analytic functions as follows:

Definition 2.3 *Let p* ∈ P, *such that* $p(z) = 1 + \sum_{k=2}^{\infty} a_k z^{k-1}$. *Then the q-derivative,*

$D_{z,q}$ *on the function f*∈ A_k *is define as follows:*

$$\begin{cases} (\mathcal{J}_{\mu}^{1,\mathcal{A}_k}p)(q;z) := (\mathcal{D}_{z,q}f)(z) = p(z) * \prod_{\mu=0}^{1} \frac{1}{(1-q^{\mu}z)}, & m = 1 \\ ; \\ (\mathcal{J}_{\mu}^{m,\mathcal{A}_k}p)(q;z) := (\mathcal{D}_{z,q}^{m}f)(z) = p(z) * \prod_{\mu=0}^{m} \frac{[\mu]_q}{(1-q^{\mu}z)}, & m \in \mathbb{N} \setminus \{1\}. \end{cases}$$

Definition 2.4 *The function f*∈ H_0 *is harmonic q-starlike function* $\mathcal{PS}_{q,\mathcal{H}}^{*}$ *if*

$$\frac{\partial}{\partial \phi} \arg\left[\mathcal{D}_{z,q}^{1;\mathcal{H}} f(r\exp[i\phi])\right]\Big) \geq 0, \quad z = r\exp[i\phi], \quad \phi \in [0, 2\pi], \quad r \in (0,1), \quad (2.3)$$

or equivalently

(2.4) $$\mathcal{R}e\left\{\frac{(\mathcal{D}_{z,q}^{1;\mathcal{H}}f)(z)}{f(z)}\right\} > 0,$$

where

. (2.5) $$(\mathcal{D}_{z,q}^{1;\mathcal{H}}f)(z) := z(\mathcal{D}_{z,q}h)(z) - \overline{z(\mathcal{D}_{z,q}g)(z)}$$

Remark 2.1 *Assume*

$$\frac{(\mathcal{D}_{z,q}^{1;\mathcal{H}}f)(z)}{f(z)} = \frac{z(\mathcal{D}_{z,q}f)(z)}{f(z)}$$

and substituting $\frac{(\mathcal{D}_{z,q}^{1;\mathcal{H}}f)(z)}{f(z)}$ *for* $\frac{z(\mathcal{D}_{z,q}f)(z)}{f(z)}$ *in (1.1) gives (2.4).*

We introduce Janowski *q*-classes of functions as follow:

Let −A ≤ E <A ≤ 1, then,

- $\mathcal{PS}_{q,\mathcal{H}}^{*}(\mathcal{E},\mathcal{A})$ denote the class functions *f* ∈ S_H such that

$$\mathcal{R}e\left\{\frac{(\mathcal{D}_{z,q}^{1;\mathcal{H}}f)(z)}{f(z)}\right\} \prec \frac{1+\mathcal{E}z}{1+\mathcal{A}z}. \quad (2.6)$$

- and $\mathcal{PS}_{q,\mathcal{P}}^{*}(\mathcal{E},\mathcal{A})$ denote the class functions *p* ∈ P such that

$$\frac{z(\mathcal{D}_{z,q}p)(z)}{p(z)} \prec \frac{1+\mathcal{E}z}{1+\mathcal{A}z}. \quad (2.7)$$

Necessary and Sufficient Conditions

Using the technique of Dziok [25] we, calculate the necessary and sufficient conditions for the function *f*∈ S_H to be in the class $\mathcal{PS}_{q,\mathcal{H}}^{*}(\mathcal{E},\mathcal{A})$.

Theorem 3.1 *Let f* ∈ S_H, *then* $f \in \mathcal{PS}_{q,\mathcal{H}}^{*}(\mathcal{E},\mathcal{A})$ *if and only if*

$f(z) * \Phi(z;\varsigma) \neq 0, \qquad (\varsigma \in \mathbb{C}, |\varsigma| = 1) \qquad (3.1)$

where

$$\Phi(z;\varsigma) = \frac{(\mathcal{A}-\mathcal{E})\varsigma z + (1+\mathcal{E}\varsigma)qz^2}{1-[2]_q z + qz^2} - \frac{[2+(\mathcal{E}+\mathcal{A})\varsigma]\,\overline{z} - (1+\mathcal{E}\varsigma)q\overline{z}^2}{1-[2]_q \overline{z} + q\overline{z}^2}.$$

Proof. Let $f \in S_H$ of the form (2.1). Then $f \in \mathcal{PS}^{*}_{q;\mathcal{H}}(\mathcal{E},\mathcal{A})$ if and only if equation

(2.6) is satisfied, or equivalently

$$\mathcal{R}e\left\{\frac{(\mathcal{D}^{1,\mathcal{H}}_{z,q}f)(z)}{f(z)}\right\} - \frac{1+\mathcal{E}\varsigma}{1+\mathcal{A}\varsigma} \neq 0, \qquad (\varsigma \in \mathbb{C},\ |\varsigma| = 1) \qquad (3.2)$$

To prove Theorem 3.1, we need to show that conditions (3.1) and (2.6) are equivalent. Since

$$z(\mathcal{J}^{1,\mathcal{H}}_{\mu}h)(q;z) = z(\mathcal{D}_{z,q}h)(z) = h(z) * \frac{z}{1-[2]_q z + qz^2} \qquad (3.3)$$

and

$$h(z) = h(z) * \frac{z}{1-z}.$$

From (3.2), we obtain

$$\begin{aligned}
(1+\mathcal{A}\varsigma)(\mathcal{D}^{1,\mathcal{H}}_{z,q}f)(z) - (1+\mathcal{E}\varsigma)f(z)
&= (1+\mathcal{A}\varsigma)z(\mathcal{D}_{z,q}h)(z) - (1+\mathcal{E}\varsigma)h(z) \\
&\quad - \left[(1+\mathcal{A}\varsigma)\overline{z(\mathcal{D}_{z,q}g)(z)} - (1+\mathcal{E}\varsigma)\overline{g(z)}\right] \\
&= h(z) * \left(\frac{z(1-\mathcal{A}\varsigma)}{1-[2]_q z + qz^2} - \frac{z(1+\mathcal{E}\varsigma)}{1-z}\right)
\end{aligned}$$

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$$\begin{aligned}
&- \overline{g(z)} * \left(\frac{\overline{z}(1+\mathcal{A}\varsigma)}{1-[2]_q\overline{z} + q\overline{z}^2} + \frac{\overline{z}(1+\mathcal{E}\varsigma)}{1-\overline{z}}\right) \\
&= f(z) * \Phi(z;\varsigma). \qquad (3.4)
\end{aligned}$$

Hence, the proof is complete.

Substituting E = −A = −1 in Theorem 3.1 gives the next Corollary:

Corollary 3.2 *Let* E = −A = −1 *with* $f \in S_H$. *Then* $f \in \mathcal{PS}^{*}_{q;\mathcal{H}}(-1,1)$ *if and only if*

$f(z) * \Phi(z;\varsigma) \neq 0, \qquad (\varsigma \in \mathbb{C}, |\varsigma| = 1) \qquad (3.5)$

where

$$\Phi(z;\varsigma) = \frac{2\varsigma z + (1-\varsigma)qz^2}{1-[2]_q z + qz^2} - \frac{2\overline{z} - (1-\varsigma)q\overline{z}^2}{1-[2]_q\overline{z} + q\overline{z}^2}.$$

Next, we show the sufficient condition for function $f \in H_0$ to be in the class

$\mathcal{PS}^*_{q,\mathcal{H}}(\mathcal{E},\mathcal{A})$.

Theorem 3.3 *Let $f \in H_0$. If f is defined as (2.2) and satisfies the condition*

$$\sum_{k=1}^{\infty} (\lambda_k|a_k| + \tau_k|b_k|) \le 2(\mathcal{A}-\mathcal{E}), \quad (3.6)$$

where

$\lambda_k + (1 + E) = [k]_q(1 + A)$

and

$$\tau_k - (1 + E) = [k]_q(1 + A). \quad (3.7)$$

Then $f \in \mathcal{PS}^*_{q,\mathcal{H}}(\mathcal{E},\mathcal{A})$.

Proof. Let $f(z) \equiv z$ then Theorem 3.3 is true. If $f \in H_0$ of the form (2.2) and there exist $k \in N$ such that $a_k 6= 0$, $k \ge 2$ with $b_k 6= 0$, since

$$\lambda_k \ge [k]_q(A - E), \quad \tau_k \ge [k]_q(A - E), \quad k \in N,$$

then (3.6) gives

$$\sum_{k=2}^{\infty}[k]_q(|a_k| + |b_k|) \le 1 - |b_1|$$

and

$$|\mathcal{D}_{z,q}h(z)| - |\mathcal{D}_{z,q}g(z)| \ge 1 - b_1 - \sum_{k=2}^{\infty}[k]_q(|a_k| + |b_k|)|z|^{k-1}$$

$$\ge 1 - b_1 - |z|\sum_{k=2}^{\infty}[k]_q(|a_k| + |b_k|)(1-b_1)(1-|z|)$$

$$\ge 0, \quad z \in \mathcal{U}. \quad (3.8)$$

Since $k > [k]_q$ for all $k \in N \setminus \{1\}$, $q \in (0,1)$ then,

$|h^0(z)| - |g^0(z)| \ge |D_{z,q}h(z)| - |D_{z,q}g(z)| \ge 0$

and

$|h^0(z)| - |g^0(z)| \ge 0.$

Hence, f is locally univalent and sense-preserving in U.

Suppose g is identically zero , Ismail et. al. [7] (see also [8]) proved the univalency of the function $f \in \mathcal{PS}^*_q$. Assume g is not equivalent to zero, we need to show that z_1 6= z_2 whenever $f(z_1) \neq f(z_2)$. If z_1, $z_2 \in$ U with $z_1 - z_2 \neq 0$, since U is simply connected and convex, then

$$z(\zeta) = (1-\zeta)z_1 + \zeta z_2 \in U, \quad \leq \zeta \leq 1.$$

Hence, we can write

$$|f(z_2) - f(z_1)| = \int_0^1 \left[(z_2 - z_1)(D_{z,q}h)\,(z(\zeta)) + \overline{(z_2 - z_1)\left(D_{z,q}g(\zeta)\right)}\,\right] d_q\zeta\,. \qquad (3.9)$$

Taking the real parts of equation (3.9) divided by $(z_2 - z_1)$ gives,

$$\mathcal{R}e\left\{\frac{f(z_2) - f(z_1)}{z_2 - z_1}\right\} = \int_0^1 \mathcal{R}e\left\{(\mathcal{D}_{z,q}h)(z(\zeta)) + \frac{\overline{(z_2 - z_1)}}{(z_2 - z_1)}\overline{(\mathcal{D}_{z,q}g)(\zeta)}\right\} d_q\zeta.$$

$$> \int_0^1 \mathcal{R}e\left\{(\mathcal{D}_{z,q}h)(z(\zeta)) - |(\mathcal{D}_{z,q}g)(\zeta)|\right\} d_q\zeta. \qquad (3.10)$$

That means

$$\mathcal{R}e\left\{(\mathcal{D}_{z,q}h)(z(\zeta))\right\} - |(\mathcal{D}_{z,q}g)(\zeta)| \geq \mathcal{R}e\left\{(\mathcal{D}_{z,q}h)(z(\zeta))\right\} - \sum_{k-1}^{\infty}[k]_q|b_k|$$

$$\geq 1 - \sum_{k=2}^{\infty}[k]_q|a_k| - \sum_{k=1}^{\infty}[k]_q|b_k|$$

$$\geq 1 - \sum_{k=2}^{\infty}k|a_k| - \sum_{k=1}^{\infty}k|b_k|.$$

And

$$\mathcal{R}e\left\{(\mathcal{D}_{z,q}h)(z(\zeta))\right\} - |(\mathcal{D}_{z,q}g)(\zeta)| \geq 0 \qquad (3.11)$$

also

$$\mathcal{R}e\left\{\frac{f(z_2) - f(z_1)}{z_2 - z_1}\right\} \geq 0. \qquad (3.12)$$

Hence, f is univalent.

Therefore, $f \in \mathcal{PS}^*_{q,\mathcal{H}}(\mathcal{E}, \mathcal{A})$ if and only if there exits a complex-valued function ν, with $\nu(0) = 0$, and $|\nu(z)| < 1$, $z \in$ U, such that

$$\frac{(\mathcal{D}^{1;\mathcal{H}}_{z,q}f)(z)}{f(z)} = \frac{1 + \mathcal{E}\nu(z)}{1 + \mathcal{A}\nu(z)}, \qquad (3.13)$$

that is

$$\left|\frac{(\mathcal{D}^{1;\mathcal{H}}_{z,q}f)(z) - f(z)}{\mathcal{A}(\mathcal{D}^{1;\mathcal{H}}_{z,q}f)(z - \mathcal{E}f(z)}\right| < 1, \quad z \in \mathcal{U} \qquad (3.14)$$

We need to show that

$$\left|(\mathcal{D}_{z,q}^{1,\mathcal{H}}f)(z)-f(z)\right|-\left|\mathcal{A}(\mathcal{D}_{z,q}^{1,\mathcal{H}}f)(z)-\mathcal{E}f(z)\right|<0,\quad z\in\mathcal{U}\setminus\{0\}. \tag{3.15}$$

Let $r \in (0,1)$ be the radius of U, we have

$$\left|(\mathcal{D}_{z,q}^{1,\mathcal{H}}f)(z)-f(z)\right|-\left|\mathcal{A}(\mathcal{D}_{z,q}^{1,\mathcal{H}}f)(z)-\mathcal{E}f(z)\right|$$

$$=\left|\sum_{k=2}^{\infty}([k]_q-1)a_kz^k-\sum_{k=1}^{\infty}([k]_q+1)\overline{b_k}\ \overline{z^k}\right|$$

$$-\left|(\mathcal{A}-\mathcal{E})z+\sum_{k=2}^{\infty}(\mathcal{A}[k]_q-\mathcal{E})a_kz^k-\sum_{k=1}^{\infty}(\mathcal{A}[k]_q+\mathcal{E})\overline{b_k}\ \overline{z^k}\right|$$

$$\leq\sum_{k=2}^{\infty}([k]_q-1)\,|a_k|\,r^k+\sum_{k=1}^{\infty}([k]_q+1)\,|b_k|\,r^k$$

$$-(A-\mathcal{E})r+\sum_{k=2}^{\infty}\left(A\,[k]_q-\mathcal{E}\right)|a_k|\,r^k-\sum_{k=1}^{\infty}\left(A\,|k|_q+\mathcal{E}\right)|b_k|r^k$$

$$\leq r\left\{\sum_{k=1}^{\infty}(\lambda_k|a_k|b_k)r^{k-1}-2(A-\mathcal{E})\right\}<0\,.$$

Hence,

$$f\in\mathcal{PS}_{q,\mathcal{H}}^{*}(\mathcal{E},\mathcal{A}).$$

In 1975, Silverman [22] studied the harmonic univalent functions $f \in H_0$ with negative coefficient of the form

$$f=h+\overline{g},\ h(z)=z-\sum_{k=2}^{\infty}|a_k|z^k,\quad g(z)=(-)^{\delta}\sum_{k=2}^{\infty}|b_k|z^k,\ \delta\in[0,1) \tag{3.16}$$

where $a_k = -|a_k|$, $b_k = (-1)^{\delta}|b_k|$, $k \in \mathbb{N} \setminus \{1\}$. We denote such class of functions defined in equation (3.16) by $P\mathfrak{J}^{\delta}$.

Next, we calculate the necessary and sufficient for the function $f \in P\mathfrak{J}^0$ to be in the class of function $\mathcal{PS}_{q,\mathfrak{J}}^{*}(\mathcal{E},\mathcal{A})$

Theorem 3.4 *Let* $\mathcal{PS}_{q,\mathfrak{J}}^{*}(\mathcal{E},\mathcal{A}) := \mathcal{P}\mathfrak{J}^0\cap\mathcal{PS}_{q,\mathcal{H}}^{*}(\mathcal{E},\mathcal{A})$ *with* $f \in PJ^0$. *Then* $f \in \mathcal{PS}_{q,\mathfrak{J}}^{*}(\mathcal{E},\mathcal{A})$ *if and only if*

$\sum_{k=1}^{\infty}(\lambda_k\,|a_k|+\tau_k\,|b_k|)\leq 2(A-\mathcal{E}),$ (3.17)

where

$\lambda_k + (1 + \mathcal{E}) = [k]_q(1 + \mathcal{E})$

and

$\tau_k - (1 + \mathcal{E}) = [k]_q(1 + \mathcal{E}).$ (3.18)

Proof. The function $f \in \mathcal{PS}^*_{q,\mathfrak{J}}(\mathcal{E}, \mathcal{A})$ if and only if

$$\left| \frac{(\mathcal{D}^{1,\mathcal{H}}_{z,q} f)(z) - f(z)}{\mathcal{A}(\mathcal{D}^{1,\mathcal{H}}_{z,q} f)(z - \mathcal{E} f(z)} \right| < 1, \quad z \in \mathcal{U}$$

which is equivalent to the following equation:

$$\left| \frac{\sum_{k=1}^{\infty} \left\{ ([k]_q - 1)|a_k| \ |z|^k + ([k]_q + 1)|b_k| \ |\overline{z}|^k \right\}}{2(\mathcal{A} - \mathcal{E})z - \sum_{k=1}^{\infty} \left\{ (\mathcal{A}[k]_q - \mathcal{E})|a_k| z^k + (\mathcal{A}[k]_q + \mathcal{E})|b_k| \ \overline{z^k} \right\}} \right| < 1 . \qquad (3.19)$$

Let $r \in (0,1)$ be the radius of U, with $z \in$ U. We have

$$\frac{\sum_{k=1}^{\infty} \left\{ ([k]_q - 1)|a_k| + ([k]_q + 1)|b_k| \right\} r^{k-1}}{2(\mathcal{A} - \mathcal{E}) - \sum_{k=1}^{\infty} \left\{ (\mathcal{A}[k]_q - \mathcal{E})|a_k| + (\mathcal{A}[k]_q + \mathcal{E})|b_k| \right\} r^{k-1}} < 1 . \quad (3.20)$$

The numerator of (3.20) cannot vanish for $0 < r < 1$. Thus

$$\sum_{k=1}^{\infty} \left\{ (\mathcal{A}[k]_q - \mathcal{E})|a_k| + (\mathcal{A}[k]_q + \mathcal{E})|b_k| \right\} r^{k-1} < 2(\mathcal{A} - \mathcal{E}) .$$

Applying the method of proving Theorem 3.3, the theorem has been proved. Next,we calculate some subordinate property of the class $\mathcal{PS}^*_{q,\mathcal{J}}(\mathcal{E}, \mathcal{A})$.

The Subordination Theory of the analytic Class of Janowski Functions

In this section we assume g is equivalent to zero. Using the technique of [30] we have the following subordination results.

Lemma 4.1 *Let h be starlike in* U*. Let* $\phi(0) = a$*,* $a \in$ R+ *with*

$$z(\mathrm{D}_{(z,q)}\phi)(z) \prec h(z). \qquad (4.1)$$

Then

$$\varphi(z) \prec a - \int_0^z h(\zeta)\zeta^{-1} d_{(\zeta,q)}(\zeta) .$$

Proof. Equation (4.6) is equivalent to

$$(\mathrm{D}_{(z,q)}\phi)(z) \prec z^{-1}h(z). \qquad (4.2)$$

Calculating the q-integral on both sides of (4.2) from 0 to z gives

$$\varphi(z) \prec a - \int_0^z h(\zeta)\zeta^{-1} d_{(\zeta,q)}(\zeta) .$$

Next,

Corollary 4.2 *Let h be starlike in* U*. Let* $\phi \in$ P *with*

$$\frac{z(\mathcal{D}_{(z,q)}\varphi)(z)}{\varphi(z)} \prec h(z). \quad (4.3)$$

Then

$$\varphi(z) \prec \exp\left\{\frac{\ln q}{(q-1)}\left[\int_0^z h(\zeta)\zeta^{-1}d_{(\zeta,q)}(\zeta)\right]\right\}.$$

Proof. Let $P(z) = Log\phi(z)$, then we can rewrite (4.6) as

$$z\left(\mathcal{D}_{(z,q)}P\right)(z) \prec h(z)$$

equivalently as

$$\left(\mathcal{D}_{(z,q)}P\right)(z) \prec z^{-1}h(z). \quad (4.4)$$

Calculating the q-integral on both sides of (4.4) and applying Lemma 1.1 gives

$$\frac{(1-q)}{\ln q}\text{Log } \varphi(z) \prec \int_0^z \frac{h(\zeta)}{\zeta}d_{(\zeta,q)}(\zeta). \quad (4.5)$$

Equation (4.5) gives the required result.

Next,

Corollary 4.3 *Assume* $h(z) = \frac{1+\mathcal{E}z}{1+\mathcal{A}z}$, *and* $\phi \in$ P *with* $\varphi \in \mathcal{PS}^*_{q,\mathcal{P}}(\mathcal{E},\mathcal{A})$. *Then*

$$\varphi(z) \prec \exp\left\{\frac{\ln q}{(q-1)}\left[\int_0^z \frac{1}{\zeta}\left(\frac{1+\mathcal{E}\zeta}{1+\mathcal{A}\zeta}\right)d_{(\zeta,q)}(\zeta)\right]\right\}.$$

Proof. Let $\varphi \in \mathcal{PS}^*_{q,\mathcal{P}}(\mathcal{E},\mathcal{A})$ then,

$$\frac{z(\mathcal{D}_{(z,q)}\varphi)(z)}{\varphi(z)} \prec \frac{(1+\mathcal{E}z)}{1+\mathcal{A}z}. \quad (4.6)$$

The q-integral of equation (4.6) gives the required result.

Next,

Corollary 4.4 *Assume* E = q *and* A = $-q^2$ *and* $\varphi(z) \in \mathcal{PS}^*_{q,\mathcal{P}}(\mathcal{E},\mathcal{A})$.

Then,

$$\varphi(z) \prec \left[\frac{z}{1-[2]_q z + qz^2}\right]^{\sigma}$$

where $\sigma = \frac{\ln q}{1-q}$.

Proof. Let

$$\mathcal{K}_q = \frac{z}{1-[2]_q z + qz^2} \quad (4.7)$$

Simple calculation on (4.7) gives

$$\frac{z(\mathcal{D}_{z,q}\mathcal{K}_q)(z)}{\mathcal{K}_q} = \frac{(1+qz)}{(1-q^2z)}. \quad (4.8)$$

Hence (4.6) can be rewritten as

$$\frac{(\mathcal{D}_{(z,q)}\varphi)(z)}{\varphi(z)} \prec \frac{(\mathcal{D}_{z,q}\mathcal{K}_q)(z)}{\mathcal{K}_q}. \quad (4.9)$$

Calculating q-integral of (4.9) gives the required result.

Results and Discussion

Firstly, we present the results in this article as follows:

Let $f \in S_H$, we calculated the necessary and sufficient conditions for the function f to be in the class of $\mathcal{PS}^*_{q,\mathcal{H}}(\mathcal{E},\mathcal{A})$. Suppose, $f \in H_0$, we calculated the sufficient condition for f to be in the class of function $\mathcal{PS}^*_{q,\mathcal{H}}(\mathcal{E},\mathcal{A})$. If $f \in \mathcal{P}\mathfrak{J}^0$, we calculated the necessary and sufficient conditions for f to be in the class of function $\mathcal{PS}^*_{q,\mathfrak{J}}(\mathcal{E},\mathcal{A})$.

Furthermore, with the assumption that g is identically zero, we calculated the some subordination properties of q-class of functions.

Discussion

We note that

$$\lim_{q\to 1} \mathcal{PS}^*_{q,\mathcal{H}}(\mathcal{E},\mathcal{A}) = \mathcal{S}^*_{\mathcal{H}}(\mathcal{E},\mathcal{A}),$$

hence the class of function studied by [27], [25] and [26], also

$$\lim_{q\to 1} \mathcal{PS}^*_{q,\mathcal{J}}(\mathcal{E},\mathcal{A}) = \mathcal{S}^*(\mathcal{E},\mathcal{A})$$

the class of function studied by [22], [21], [24], [27], [25], [23] and [28] to mention but a few.

Due to the speculated expansion of the q-theory in geometric function theory, we introduce Lemma 4.1 and their corollaries. Lemma 4.1 is the q-class of the result of Suffridge,[29] (see also [30].)

2. Direction for further research

In conclusions, we extended the q-theory to some harmonic and analytic properties of Janowski functions. In view of the recent applications of q-theory in Geometric Function Theory, it seem natural to investigate some properties of a q-Koebe function given by(4.7).

Let $K_q$define by (4.7) be represented as

$K_q(z) = z + \sum_{k=2}^{\infty} c_k z^k$

Then, we may have the following:

$\mathcal{K}_q(z) \in \mathcal{PS}^*_q,$

$K_q(z) \in PC_q$, iii. $c_k \leq [k]_q$,

$\lim_{q\to 1} \mathcal{K}_q(z) = \frac{z}{(1-z)}$,

$\lim_{q\to 1}[k]_q = k$, where (iii) - (v) are too obvious.

In view of Clunie and Sheil-Small [21], we state an open problem for harmonic q-Koebe type of function given as follows:

Definition 6.5 *Let the function* $\varphi(z) = \frac{z}{1-[2]_q z+qz^2}$ *and*

(6.1)
$$h(z) - g(z) = \frac{z}{1-[2]_q z+qz^2} \quad (i)$$
$$(\mathcal{D}_{z,q}g)(z) - q^3 z(\mathcal{D}_{z,q}h)(z) = 0 \quad (ii)$$

.

Solving (6.1) gives

$$(\mathcal{D}_{z,q}h)(z) = \frac{1+qz}{\prod_{k=0}^{3}(1-zq^k)}, \quad (\mathcal{D}_{z,q}g)(z) = q^3 z\frac{1+qz}{\prod_{k=0}^{3}(1-zq^k)}. \quad (6.2)$$

Hence the harmonic q-Koebe functions h and g can be derived from (6.2).

3. Conclusion

The q-derivative operator was stated as a convolution of two analytic functions. The necessary and sufficient conditions for a Janowski's harmonic q-starlike functions were studied. Also, some subordination properties of Janowski's analytic q-starlike functions were studied.

Compliance with ethical standards

Disclosure of conflict of interest

The author disclosed there is no conflict of interest.

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