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On class (BD) Operators

Wanjala Victor * and Beatrice Adhiambo Obiero

Department of Mathematics and computing, Rongo University, Kitere Hills Kenya.

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Abstract

In this paper, we introduce the class of (BD) operators acting on a complex Hilbert space H. An operator if T ∈ B (H) is said to belong to class (BD) if T * 2 (TD) 2 commutes with (T *TD) 2 equivalently [T * 2 (TD) 2, (T *TD) 2] = 0. We investigate the properties of this class and we also analyze the relation of this class to D-operator and then generalize it to class (nBD) and analyze its relation to the class of n-power D-operator through complex symmetric operators.

Keywords: D-operator; Normal; N Quasi D-operator; Complex symmetric operators; N-power D-operator; (BD) Operators

1. Introduction

Throughout this paper, H denotes the usual Hilbert space over the complex field and B (H) the Banach algebra of all bounded linear algebra on an infinite dimensional separable Hilbert space H. A bounded linear operator T is said to be in class (Q) if $T^{*2}T^{2} = (T^{*}T)^{2}$ (2). This was later extended into other classes like class (Q) (2), n-power class (Q) if $T^{*2}T^{2n} = (T^{*}T^{n})^{2}$ (3), quasi-M class (Q) and (α, β)-class (Q) we refer the reader to (6) for more. An operator T ∈B (H) is said to belong to class (BQ) if $T^{*2}T^{2}(T^{*}T)^{2} = (T^{*}T)^{2}T^{*2}T^{2}$ an operator T ∈ B (H) is said to be D-operator if $T^{2}(T^{D})^{2} = (T^{2}T^{D})^{2}$ where T^{D} is the Drazin inverse of T (1). Wanjala Victor and A.M. Nyongesa later extended this to N Quasi D-operator (3), a bounded linear operator T is said to be N Quasi D-operator if $T(T^{2}(T^{D})^{2}) = N(T^{2}T^{D})^{2}T$ where N is a bounded linear operator. A bounded linear operator T is said to belong to class (BD) provided $T^{2}(T^{D})^{2}$ commutes with $(T^{2}T^{D})^{2}$ where T^{D} is the Drazin inverse of T. Let H be a Hilbert space, then a conjugation on H is an anti-linear operator C from H onto itself such that the following is satisfied Cξ, Cζi = hζ, ξi for every ξ, ζ ∈ H and $C^{2} = I$. We say that T is complex symmetric if $T = CT^{*}C$.

2. Main results

2.1. Theorem 1

Let T ∈ B (H) be such that T ∈ (BD), then the following are also true for (BD)

(i). λT for any real λ

(ii). Any S ∈ B (H) that is unitarily equivalent to T.

(iii). the restriction T/ M to any closed subspace M of H.

Proof. (i). the proof is trivial.

* Corresponding author: Wanjala Victor
Department of Mathematics and computing, Rongo University, Kitere Hills Kenya.

(ii). Let $S \in B(H)$ be unitarily equivalent to T, then there exists a unitary operator U

$\in B(H)$ with

$S = U^*TU$ and $S^* = U^*T^*U$. Since $T \in (BD)$, we have;

$T^{*2}(T^D)^2(T^*T^D)^2 = (T^*T^D)^2T^{*2}(T^D)^2$, hence

$S^{*2}(S^D)^2(S^*S^D)^2 = UT^{*2}U^*U(T^D)^2U^*(UT^*U^*UT^DU^*)^2$

$= UT^{*2}U^*U^*(T^D)^2U^*UT^*U^*UT^*U^*UT^DU^*UT^DU^*$

$= UT^{*2}(T^D)^2(T^*T^D)^2U^*$

$= U(T^*T^D)^2T^{*2}(T^D)^2U^*$

And

$(S^*S^D)^2S^{*2}(S^D)^2 = (UT^*U^*UT^DU^*)^2UT^{*2}U^*U(T^D)^2U^*$

$= UT^*U^*U(T^D)U^*UT^*U^*UT^DU^*UT^{*2}U^*U(T^D)^2U^*$

$= UT^*T^DT^*T^DT*^2T^{*2}U^*$

$= U(T^*T^D)^2T^{*2}(T^D)^2U^*$

Hence S is unitarily equivalent to T.

(iii). If T is in class (BD), then;

$T^{*2}(T^D)^2(T^*T^D)^2 = (T^*T^D)^2T^{*2}(T^D)^2$.

Hence;

$(T/M)^{*2}((T/M)^D)^2\{(T/M)^*(T/M)^D\}^2$

$= (T/M)^{*2}((T/M)^D)^2\{(T/M)^*(T/M)^D\}^2$

$= (T^{*2}/M)((T^D)^2/M)\{(T^*/M)(T^D/M)\}\{(T^*/M)(T^D/M)\}$

$= \{(T^*T^D)^2/M\}\{T^{*2}(T^D)^2/M\}$

$= \{(T^*/M)(T^D/M)\}^2(T/M)^{*2}((T/M)^D)^2$

Hence $T/M \in (BD)$.

2.2. Theorem 2

If $T \in B(H)$ is a D-operator, then $T \in (BD)$.

Proof. Suppose T is a D-operator, then

$T^{*2}(T^D)^2 = (T^*T)^2$

Post multiplying both sides by $T^{*2}(T^D)^2$;

$T^{*2}(T^D)^2T^{*2}(T^D)^2 = (T^*T^D)^2T^{*2}(T^D)^2$

$T^{*2}(T^D)^2T^*T^DT^*T^D = (T^*T^D)^2T^{*2}(T^D)^2$

$T^{*2} (T^D)^2 (T^*T^D)^2 = (T^*T^D)^2 T^{*2} (T^D)^2$.

2.3. Theorem 3

Let $S \in (BD)$ and $T \in (BD)$. If both S and T are doubly commuting, then

ST is in (BD).

Proof.

$(ST)^{*2} ((ST)^D)^2 ((ST)^* (ST)^D)^2$

$= S^{*2} T^{*2} (S^D)^2 (T^D)^2 ((ST)^* (ST)^D) ((ST)^* (ST)^D)$

$= S^{*2} T^{*2} (S^D)^2 (T^D)^2 ((S^*T^*) (ST)^D) ((S^*T^*) (ST)^D)$

$= S^{*2} T^{*2} (S^D)^2 (T^D)^2 S^* T^* S^D T^D S^* T^* S^D T^D S^* T^* S^D T^D$

$= S*^2 T*^2 (SD)^2 (TD)^2 S* (SD) T* (T D) S* (SD) T* TD$

$= T^{*2} (T^D)^2 S^{*2} (S^D)^2 S^* S^D S^* S^D T^* T^D T^* T^D$

$= T^{*2} (T^D)^2 S^{*2} (S^D)^2 (S^*S^D)^2 T^* T^D T^* T^D$

$= T^{*2} (T^D)^2 (S^*S^D)^2 S^{*2} (S^D)^2 T^* T^D T^* T^D$ (Since $S \in (BD)$).

$= (S^*S^D)^2 T^{*2} (T^D)^2 T^* T^D T^* T^D S^{*2} (S^D)^2$

$= (S^*S^D)^2 T^{*2} (T^D)^2 (T^*T^D)^2 S^{*2} (S^D)^2$

$= (S^*S^D)^2 (T^*T^D)^2 T^{*2} (T^D)^2 S^{*2} (S^D)^2$ (Since $T \in (BD)$).

$= ((S^*S^D) (T^*T^D))^2 T^{*2} S^{*2} (T^D)^2 (S^D)^2$

$= ((S^*T^*) (S^D T^D))^2 S^{*2} T^{*2} (S^D)^2 (T^D)^2$

$= ((ST)^* (ST)^D)^2 (ST)^{*2} ((ST)^D)^2$

Hence $ST \in (BD)$.

2.4. Theorem 4

Let $T \in B(H)$ be a class (BD) operator such that $T = CT^*C$ with C being a

Conjugation on H. If C is such that it commutes with $T^{*2} (T^D)^2$ and $(T^*T^D)^2$, then T is a

D- Operator.

Proof. Let $T \in (BD)$ and complex symmetric, then we have; $T^{*2} (T^D)^2 (T^*T^D)^2 = (T^*T^D)^2 T^{*2} (T^D)^2$

And $T = CT^*C$.

Hence;

$T^{*2} (T^D)^2 (T^*T^D)^2 = (T^*T^D)^2 T^{*2} (T^D)^2$

$T^{*2} (T^D)^2 CT^D CCT^* CCT^D CCT^* C = (T^*T^D)^2 CT^D CCT^* CCT^D CCT^* C$.

$T^{*2} (T^D)^2 CT^D T^* T^D T^* C = (T^*T^D)^2 CT^D T^* T^D T^* C$

$T^{*2} (T^D)^2 C (T^D)^2 T^{*2} C = (T^*T^D)^2 CT^*T^DT^*T^DC$

$T^{*2} (T^D)^2 CT^{*2} (T^D)^2 C = (T^*T^D)^2 C (T^*T^D)^2 C.$

C commutes with $T^{*2} (T^D)^2$ and $(T^*T^D)^2$ hence we obtain;

$T^{*2} (T^D)^2 T^{*2} (T^D)^2 = (T^*T^D)^2 (T^*T^D)^2.$

Which implies;

$T^{*2} (T^D)^2 = (T^*T^D)^2$ and hence T is a D-operator.

Definition 5. An operator T is said to be in class (nBD) if $T^{*2} (T^D)^{2n} (T^* (T^D)^n)^2 = (T^* (T^D)^n)^2 T^{*2} (T^D)^{2n}$ for a positive integer n.

2.5. Theorem 6

Let $T \in B(H)$ be (n-1)-D- operator, if T is a complex symmetric

Operator such that C commutes with $(T^* T^D)^2$, then T is an n-power D- operator.

Proof. With T being complex symmetric and (n-1)-D-operator, we have;

$T = CT^*C$ and $T^{*2} (T^D)^{2n-2} = (T^* (T^D)^{n-1})^2.$

We obtain;

$T^{*2} (T^D)^{2n-2} (T^D)^2 = (T^* (T^D)^{n-1})^2 (T^D)^2.$

Hence;

$T^{*2} (T^D)^{2n} = (T^* (T^D)^{n-1})^2 (T^D)^2.$

$T^{*2} (T^D)^{2n} = T^{*2} (T^D)^{2n-2} (T^D)^2 = (T^D)^{2n-2} T^{*2} (T^D)^2$

$T^{*2} (T^D)^{2n} = (T^D)^{2n-2} T^*T^*T^DT^D = (T^D)^{2n-2} CT^DCCT^DCCT^*CCT^*C = (T^D)^{2n-2} CT^DT^DT^*T^*C.$

$=T^{*2} (T^D)^{2n} = (T^D)^{2n-2} C (T^D)^2 T^{*2} C = (T^D)^{2n-2} C (T^*T^D)^2 C$

Since C commutes with $(T^*T^D)^2$ we obtain;

$T^{*2} (T^D)^{2n} = (T^D)^{2n-2} (T^*T^D)^2 CC = (T^D)^{2n-2} T^{*2} (T^D)^2 CC = (T^D)^{2n-2} (T^D)^2 T^{*2} CC = T^{*2} (T^D)^{2n} = (T^* (T^D)^G)^2$

Hence T is n-power D-operator

3. Conclusion

The study of class (BD) operators will help in the enhancement of study of properties of various classes such as class (Q) operators, normal operators and binormal operators.

Compliance with ethical standards

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Disclosure of conflict of interest

The authors declared no conflict of interest.

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