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Optimization of one-step hybrid method for direct solution of fifth order ordinary differential equations of initial value problems

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Abstract

This paper focuses on the derivation, analysis and implementation of a hybrid method by optimizing the order of the method by introduction of six-hybrid points for direct solution of fifth order ordinary differential equations of initial value problems (IVPs). Power series was used as the basis function for the solution of the IVP. The basis function was interpolated at some selected hybrid points whereas the fifth derivative of the approximate solution was collocated at all the interval of integration of the method to generate a system of linear equations for the determination of the unknown parameters. The derived method was tested for consistency, zero stability, convergence and absolute stability. The method was tested with two linear test problems to confirm its accuracy and usability. The comparison of the results with some existing methods shows the superiority of the accuracy of the method.

AMS Subject Classification: 65L05, 65L06, 65L10, 65L12

Keywords: Hybrid Method; Fifth Order ODEs; Initial Conditions; Linear Fifth Order Problems

1. Introduction

In this paper, we consider fifth-order ordinary differential equations, ODEs which we often encounter in the field of sciences, engineering and dynamic systems. They are generally written as

$$y^{v}(x) = f(x, y, y^{i}, y^{ii}, y^{iii} y^{iv}), \qquad (1)$$

with the following initial conditions

$$y(x_o) = y_0, y^{i}(x_o) = y_1, y^{ii}(x_o) = y_2, y^{iii}(x_o) = y_3, y^{iv}(x_o) = y_4$$

In literature, it has been shown that attention has been directed at solving higher order ordinary differential equations with Linear Multistep Methods [LMMs] or One-step method without following the conventional method of reduction to systems of first order problems, see [1-10].

The essence of this work is to avoid the inherent drawbacks associated with such methods and to improve accuracy. The many research activities along this line have produced numerical schemes that are not sufficient enough in handling fifth order problems directly without reducing them to lower order problems. [11] proposed an order six multi-derivative implicit (closed) methods was proposed to solve fifth order ordinary differential Equations. In the work of

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[12], an explicit method of order six was used as a main predictor for its implementation was presented. The setback associated with such method lies in the fact that they have low order of accuracy; hence the performances of their methods are not good enough. In this present work, we presents an optimization of one-step hybrid method with order of accuracy of eight for direct solution of fifth order ordinary differential equations for the purpose of enhancing accuracy. The proposed method is zero stable, A-stable, consistent and convergent.

2. Derivation of the Method

The exact solution $y(x)$ to (1) is approximated by form

$$y(x) = \sum_{j=0}^{c+i-1} a_j x^j \qquad (2)$$

with the fifth derivative given as

$$y^v(x) = \sum_{j=5}^{c+i-1} j(j-1)(j-2)(j-3)(j-4)a_j x^{j-5} \qquad (3)$$

where c is the number of collocation points and i is the number of interpolation points. (2) is called interpolation equation while (3) is called collocation equation.

Collocating (3) at the points $x = x_n, x_{n+a}, x_{n+b}, x_{n+c}, x_{n+d}, x_{n+e}$ and x_{n+1} and interpolating (2) at the points $x = x_{n+a}, x_{n+b}, x_{n+c}, x_{n+d}$ and x_{n+e} leads to a system of twelve equations which is solved by a Computer Aided Software such as Maple to obtain the required parameters. For simplicity, we compute $a = \frac{1}{7}, b = \frac{2}{7}, c = \frac{3}{7}, d = \frac{4}{7}, e = \frac{5}{7}$.

$$a_0 = -\frac{1873}{67765824}h^5 f_{n+\frac{3}{7}} - \frac{523}{203297472}h^5 f_{n+\frac{1}{7}} + \frac{1}{67765824}h^5 f_n + 5y_{n+\frac{1}{7}} - 10y_{n+\frac{2}{7}} + 10y_{n+\frac{3}{7}} - 5y_{n+\frac{4}{7}} + y_{n+\frac{5}{7}} - \frac{1}{203297472}h^5 f_{n+1} - \frac{11}{67765824}h^5 f_{n+\frac{5}{7}} + \frac{1}{22588608}h^5 f_{n+\frac{6}{7}} - \frac{1831}{67765824}h^5 f_{n+\frac{2}{7}} - \frac{439}{203297472}h^5 f_{n+\frac{4}{7}} \qquad (4)$$

$$a_1 = -\frac{1}{19168047360}\frac{1}{h}\begin{pmatrix} 635h^5 f_n + 467h^5 f_{n+1} \\ +1297121h^5 f_{n+\frac{1}{7}} + 8539731h^5 f_{n+\frac{2}{7}} \\ +7707593h^5 f_{n+\frac{3}{7}} + 672593h^5 f_{n+\frac{4}{7}} \\ +15075h^5 f_{n+\frac{5}{7}} - 4543h^5 f_{n+\frac{6}{7}} \\ -860964793920y_{n+\frac{1}{7}} + 23928125440y_{n+\frac{2}{7}} \\ -2616438464640y_{n+\frac{3}{7}} + 1364126037120y_{n+\frac{4}{7}} \\ -279534924000y_{n+\frac{5}{7}} \end{pmatrix} \qquad (5)$$

$$a_2 = -\frac{1}{19168047360}\frac{1}{h^2}\begin{pmatrix} 997331h^5 f_n + 24745h^5 f_{n+1} \\ +58102075h^5 f_{n+\frac{1}{7}} + 210880995h^5 f_{n+\frac{2}{7}} \\ +164992235h^5 f_{n+\frac{3}{7}} + 13625545h^5 f_{n+\frac{4}{7}} \\ +645969h^5 f_{n+\frac{5}{7}} - 204895h^5 f_{n+\frac{6}{7}} - \\ 1190814942200y_{n+\frac{1}{7}} + 39582017798400y_{n+\frac{2}{7}} - \\ 49309801833600y_{n+\frac{3}{7}} + 27506147961600y_{n+\frac{4}{7}} \\ -5870214504000y_{n+\frac{5}{7}} \end{pmatrix} \qquad (6)$$

$$a_3 = -\frac{1}{533433600}\frac{1}{h^3}\begin{pmatrix} 114677f_n + 4581h^5 f_{n+1} + 1967439h^5 f_{n+\frac{1}{7}} \\ +3067899h^5 f_{n+\frac{2}{7}} + 2544455h^5 f_{n+\frac{3}{7}} \\ -84045h^5 f_{n+\frac{4}{7}} + 133653h^5 f_{n+\frac{5}{7}} \\ -37459h^5 f_{n+\frac{6}{7}} - 106731172800y_{n+\frac{1}{7}} \\ +396430070400y_{n+\frac{2}{7}} - 548903174400y_{n+\frac{3}{7}} \\ +335440828800y_{n+\frac{4}{7}} - 76236552000y_{n+\frac{5}{7}} \end{pmatrix} \quad (7)$$

$$a_4 = -\frac{1}{101606400}\frac{1}{h^4}\begin{pmatrix} 178352f_n + 5693h^5 f_{n+1} \\ +922891h^5 f_{n+\frac{1}{7}} + 129752h^5 f_{n+\frac{2}{7}} \\ +786775h^5 f_{n+\frac{3}{7}} - 331920h^5 f_{n+\frac{4}{7}} \\ +169633h^5 f_{n+\frac{5}{7}} - 46776h^5 f_{n+\frac{6}{7}} \\ -10164873600y_{n+\frac{1}{7}} + 40659494400y_{n+\frac{2}{7}} \\ -60989241600y_{n+\frac{3}{7}} + 40659494400y_{n+\frac{4}{7}} \\ -10164873600y_{n+\frac{5}{7}} \end{pmatrix} \quad (8)$$

$$a_5 = \frac{1}{120}f_n \quad (9)$$

$$a_6 = -\frac{1}{43200}\frac{1}{h}\begin{pmatrix} 1089f_n - 60f_{n+1} - 2940f_{n+\frac{1}{7}} + 4410f_{n+\frac{2}{7}} \\ -4900f_{n+\frac{3}{7}} + 3675f_{n+\frac{4}{7}} - 1764f_{n+\frac{5}{7}} - 490f_{n+\frac{6}{7}} \end{pmatrix} \quad (10)$$

$$a_7 = -\frac{7}{43200}\frac{1}{h^2}\begin{pmatrix} 938f_n - 126f_{n+1} - 4041f_{n+\frac{1}{7}} \\ +7911f_{n+\frac{2}{7}} - 9490f_{n+\frac{3}{7}} + 7380f_{n+\frac{4}{7}} \\ -3618f_{n+\frac{5}{7}} - 1019f_{n+\frac{6}{7}} \end{pmatrix} \quad (11)$$

$$a_8 = -\frac{49}{691200}\frac{1}{h^3}\begin{pmatrix} 967f_n - 232f_{n+1} - 5104f_{n+\frac{1}{7}} \\ +11787f_{n+\frac{2}{7}} - 15560f_{n+\frac{3}{7}} + 12725f_{n+\frac{4}{7}} \\ -6432f_{n+\frac{5}{7}} + 1849f_{n+\frac{6}{7}} \end{pmatrix} \quad (12)$$

$$a_9 = -\frac{343}{311040}\frac{1}{h^4}\begin{pmatrix} 56f_n - 21f_{n+1} - 333f_{n+\frac{1}{7}} \\ +852f_{n+\frac{2}{7}} - 1219f_{n+\frac{3}{7}} + 1056f_{n+\frac{4}{7}} \\ -555f_{n+\frac{5}{7}} + 164f_{n+\frac{6}{7}} \end{pmatrix} \quad (13)$$

$$a_{10} = -\frac{2401}{3110400}\frac{1}{h^5}\begin{pmatrix} 46f_n - 25f_{n+1} - 295f_{n+\frac{1}{7}} + 810f_{n+\frac{2}{7}} \\ -1235f_{n+\frac{3}{7}} + 1130f_{n+\frac{4}{7}} - 621f_{n+\frac{5}{7}} + 190f_{n+\frac{6}{7}} \end{pmatrix} \quad (14)$$

$$a_{11} = -\frac{16807}{5702400}\frac{1}{h^6}\begin{pmatrix} 4f_n - 3f_{n+1} - 27f_{n+\frac{1}{7}} + 78f_{n+\frac{2}{7}} - 125f_{n+\frac{3}{7}} \\ +120f_{n+\frac{4}{7}} - 69f_{n+\frac{5}{7}} + 22f_{n+\frac{6}{7}} \end{pmatrix} \quad (15)$$

$$a_{12} = -\frac{117649}{60842880}\frac{1}{h^7}\begin{pmatrix} f_n - f_{n+1} - 7f_{n+\frac{1}{7}} + 21f_{n+\frac{2}{7}} - 35f_{n+\frac{3}{7}} \\ +35f_{n+\frac{4}{7}} - 21f_{n+\frac{5}{7}} + 7f_{n+\frac{6}{7}} \end{pmatrix} \quad (16)$$

Substituting $(4 - 16)$ into (2) gives a continuous coefficient of the form:

$$y(t) = \alpha_{\frac{1}{7}}(t)y_{n+\frac{1}{7}} + \alpha_{\frac{2}{7}}(t)y_{n+\frac{2}{7}} + \alpha_{\frac{3}{7}}(t)y_{n+\frac{3}{7}} + \alpha_{\frac{4}{7}}(t)y_{n+\frac{4}{7}} + \alpha_{\frac{5}{7}}(t)y_{n+\frac{5}{7}} +$$

$$h^5\left(\beta_0(t)+\beta_{\frac{1}{7}}(t)+\beta_{\frac{2}{7}}(t)+\beta_{\frac{3}{7}}(t)+\beta_{\frac{4}{7}}(t)+\beta_{\frac{5}{7}}(t)+\beta_{\frac{6}{7}}(t)+\beta_1(t)\right) \qquad (17)$$

where $\alpha_{\frac{1}{7}}(t), \alpha_{\frac{2}{7}}(t)\ldots\alpha_{\frac{5}{7}}(t)$ and $\beta_0(t), \beta_{\frac{1}{7}}(t), \ldots\beta_1(t)$ are continuous coefficients. The continuous method (17) is used to generate the required method for solving (1). That is, we evaluate at $t=1$

$$y_{n+1} = 5y_{n+\frac{1}{7}} - 24y_{n+\frac{2}{7}} + 45y_{n+\frac{3}{7}} - 40y_{n+\frac{4}{7}} + 15y_{n+\frac{5}{7}} - \frac{1}{50824368}h^5 f_n + \frac{1}{12706092}h^5 f_{n+\frac{1}{7}} + \frac{209}{16941456}h^5 f_{n+\frac{2}{7}} + \frac{3523}{254121841}h^5 f_{n+\frac{3}{7}} + \frac{8341}{50824368}h^5 f_{n+\frac{4}{7}} + \frac{83}{2117682}h^5 f_{n+\frac{5}{7}} + \frac{137}{50824368}h^5 f_{n+\frac{6}{7}} + \frac{1}{25412184}h^5 f_{n+1}$$

(18)

3. Analysis of the method

3.1. Order and error Constants of the Methods

According to ([9-17]), the order of the new method (18) is obtained by using the Taylor series. The procedure is shown below

Theorem 1: The linear operator and the associated method are said to be of order p if $C_0 = C_1 = \ldots C_p = C_{p+1} = 0, C_{p+2} = 0, C_{p+2} = 0, C_{p+4} = 0, C_{p+5} \neq 0$ C_{p+5} is called the error constant.

$$y_{n+1} = \begin{pmatrix} 5y_{n+\frac{1}{7}} - 24y_{n+\frac{2}{7}} + 45y_{n+\frac{3}{7}} - 40y_{n+\frac{4}{7}} \\ +15y_{n+\frac{5}{7}} + \frac{1}{12706092}h^5 f_{n+\frac{1}{7}} + \frac{209}{16941456}h^5 f_{n+\frac{2}{7}} \\ + \frac{3523}{25412184}h^5 f_{n+\frac{3}{7}} + \frac{8341}{50824368}h^5 f_{n+\frac{4}{7}} + \frac{83}{2117682}h^5 f_{n+\frac{5}{7}} \\ + \frac{137}{50824368}h^5 f_{n+\frac{6}{7}} - \frac{1}{25412184}h^5 f_{n+1} \end{pmatrix} \qquad (19)$$

Carrying out Taylor series expansion on (19) gives

$$\sum_{j=0}^{\infty}\frac{(1)^j(h)^j}{j!}y_n^j + 5\sum_{j=0}^{\infty}\frac{\left(\frac{1}{7}\right)^j(h)^j}{j!}y_n^j - 24\sum_{j=0}^{\infty}\frac{\left(\frac{2}{7}\right)^j(h)^j}{j!}y_n^j + 45\sum_{j=0}^{\infty}\frac{\left(\frac{3}{7}\right)^j(h)^j}{j!}y_n^j - 40\sum_{j=0}^{\infty}\frac{\left(\frac{4}{7}\right)^j(h)^j}{j!}y_n^j$$
$$+ 15\sum_{j=0}^{\infty}\frac{\left(\frac{5}{7}\right)^j(h)^j}{j!}y_n^j - \sum_{j=0}^{\infty}\frac{(h)^{j+5}}{j!}y_n^{j+5} \qquad (20) - \frac{1}{50824368}(0)^{j-5} + \frac{1}{12706092}\left(\frac{1}{7}\right)^{j-5}$$
$$+ \frac{209}{16941456}\left(\frac{2}{7}\right)^{j-5} + \frac{23523}{25412184}\left(\frac{3}{7}\right)^{j-5} + \frac{8341}{50824368}\left(\frac{4}{7}\right)^{j-5} + \frac{83}{2117682}\left(\frac{5}{7}\right)^{j-5} + \frac{137}{50824368}\left(\frac{6}{7}\right)^{j-5}$$
$$- \frac{1}{25412184}(1)^{j-5} = 0$$

Comparing the values in h gives

$$C_0 = \left(\frac{(1)^0}{0!} - \frac{1}{0!}.\left(5.\left(\frac{1}{7}\right)^0 - 24.\left(\frac{2}{7}\right)^0 + 45.\left(\frac{3}{7}\right)^0 - 40.\left(\frac{4}{7}\right)^0 + 15.\left(\frac{5}{7}\right)^0\right)\right)=0$$

$$C_1 = \left(\frac{(1)^1}{1!} - \frac{1}{1!}.\left(5.\left(\frac{1}{7}\right)^1 - 24.\left(\frac{2}{7}\right)^1 + 45.\left(\frac{3}{7}\right)^1 - 40.\left(\frac{4}{7}\right)^1 + 15.\left(\frac{5}{7}\right)^1\right)\right)=0$$

$$C_2=\left(\frac{(1)^2}{2!}-\frac{1}{0!}.\left(5.\left(\frac{1}{7}\right)^2-24.\left(\frac{2}{7}\right)^2+45.\left(\frac{3}{7}\right)^2-40.\left(\frac{4}{7}\right)^2+15.\left(\frac{5}{7}\right)^2\right)\right)=0$$

$$C_3=\left(\frac{(1)^3}{3!}-\frac{1}{3!}.\left(5.\left(\frac{1}{7}\right)^3-24.\left(\frac{2}{7}\right)^3+45.\left(\frac{3}{7}\right)^3-40.\left(\frac{4}{7}\right)^3+15.\left(\frac{5}{7}\right)^3\right)\right)=0$$

$$C_4=\left(\frac{(1)^4}{4!}-\frac{1}{4!}.\left(5.\left(\frac{1}{7}\right)^4-24.\left(\frac{2}{7}\right)^4+45.\left(\frac{3}{7}\right)^4-40.\left(\frac{4}{7}\right)^4+15.\left(\frac{5}{7}\right)^4\right)\right)=0$$

$$C_5=\left(\frac{(1)^5}{5!}-\frac{1}{5!}.\left(5.\left(\frac{1}{7}\right)^5-24.\left(\frac{2}{7}\right)^5+45.\left(\frac{3}{7}\right)^5-40.\left(\frac{4}{7}\right)^5+15.\left(\frac{5}{7}\right)^5\right)\right)-$$

$$\frac{1}{0!}\left(\begin{array}{c}-\frac{1}{50824368}(0)^0+\frac{1}{12706092}\left(\frac{1}{7}\right)^0+\frac{209}{16941456}\left(\frac{2}{7}\right)^0\\+\frac{23523}{25412184}\left(\frac{3}{7}\right)^0+\frac{8341}{50824368}\left(\frac{4}{7}\right)^0+\frac{83}{2117682}\left(\frac{5}{7}\right)^0\\+\frac{137}{50824368}\left(\frac{6}{7}\right)^0-\frac{1}{25412184}(1)^0\end{array}\right)=0$$

$$C_6=\left(\frac{(1)^6}{6!}-\frac{1}{6!}.\left(5.\left(\frac{1}{7}\right)^6-24.\left(\frac{2}{7}\right)^6+45.\left(\frac{3}{7}\right)^6-40.\left(\frac{4}{7}\right)^6+15.\left(\frac{5}{7}\right)^6\right)\right)$$

$$-\frac{1}{0!}\left(\begin{array}{c}-\frac{1}{50824368}(0)^1+\frac{1}{12706092}\left(\frac{1}{7}\right)^1+\frac{209}{16941456}\left(\frac{2}{7}\right)^1\\+\frac{23523}{25412184}\left(\frac{3}{7}\right)^1+\frac{8341}{50824368}\left(\frac{4}{7}\right)^1+\frac{83}{2117682}\left(\frac{5}{7}\right)^1\\+\frac{137}{50824368}\left(\frac{6}{7}\right)^1-\frac{1}{25412184}(1)^1\end{array}\right)=0$$

$$C_7=\left(\frac{(1)^7}{7!}-\frac{1}{7!}.\left(5.\left(\frac{1}{7}\right)^7-24.\left(\frac{2}{7}\right)^7+45.\left(\frac{3}{7}\right)^7-40.\left(\frac{4}{7}\right)^7+15.\left(\frac{5}{7}\right)^7\right)\right)$$

$$-\frac{1}{2!}\left(\begin{array}{c}-\frac{1}{50824368}(0)^2+\frac{1}{12706092}\left(\frac{1}{7}\right)^2+\frac{209}{16941456}\left(\frac{2}{7}\right)^2\\+\frac{23523}{25412184}\left(\frac{3}{7}\right)^1+\frac{8341}{50824368}\left(\frac{4}{7}\right)^2+\frac{83}{2117682}\left(\frac{5}{7}\right)^2\\+\frac{137}{50824368}\left(\frac{6}{7}\right)^2-\frac{1}{25412184}(1)^2\end{array}\right)=0$$

$$C_8=\left(\frac{(1)^8}{8!}-\frac{1}{8!}.\left(5.\left(\frac{1}{7}\right)^8-24.\left(\frac{2}{7}\right)^8+45.\left(\frac{3}{7}\right)^8-40.\left(\frac{4}{7}\right)^8+15.\left(\frac{5}{7}\right)^8\right)\right)$$

$$-\frac{1}{3!}\left(\begin{array}{c}-\frac{1}{50824368}(0)^3+\frac{1}{12706092}\left(\frac{1}{7}\right)^3+\frac{209}{16941456}\left(\frac{2}{7}\right)^3\\+\frac{23523}{25412184}\left(\frac{3}{7}\right)^3+\frac{8341}{50824368}\left(\frac{4}{7}\right)^3+\frac{83}{2117682}\left(\frac{5}{7}\right)^3\\+\frac{137}{50824368}\left(\frac{6}{7}\right)^3-\frac{1}{25412184}(1)^3\end{array}\right)=0$$

$$C_9=\left(\frac{(1)^9}{9!}-\frac{1}{9!}.\left(5.\left(\frac{1}{7}\right)^9-24.\left(\frac{2}{7}\right)^9+45.\left(\frac{3}{7}\right)^9-40.\left(\frac{4}{7}\right)^9+15.\left(\frac{5}{7}\right)^9\right)\right)$$

$$-\frac{1}{4!}\begin{pmatrix}-\frac{1}{50824368}(0)^4+\frac{1}{12706092}\left(\frac{1}{7}\right)^4+\frac{209}{16941456}\left(\frac{2}{7}\right)^4\\ +\frac{23523}{25412184}\left(\frac{3}{7}\right)^4+\frac{8341}{50824368}\left(\frac{4}{7}\right)^4+\frac{83}{2117682}\left(\frac{5}{7}\right)^4\\ +\frac{137}{50824368}\left(\frac{6}{7}\right)^4-\frac{1}{25412184}(1)^4\end{pmatrix}=0$$

$$C_{10}=\left(\frac{(1)^{10}}{10!}-\frac{1}{10!}.\begin{pmatrix}5.\left(\frac{1}{7}\right)^{10}-24.\left(\frac{2}{7}\right)^{10}\\ +45.\left(\frac{3}{7}\right)^{10}-40.\left(\frac{4}{7}\right)^{10}+15.\left(\frac{5}{7}\right)^{10}\end{pmatrix}\right)$$

$$-\frac{1}{5!}\begin{pmatrix}-\frac{1}{50824368}(0)^5+\frac{1}{12706092}\left(\frac{1}{7}\right)^5+\frac{209}{16941456}\left(\frac{2}{7}\right)^5\\ +\frac{23523}{25412184}\left(\frac{3}{7}\right)^5+\frac{8341}{50824368}\left(\frac{4}{7}\right)^5+\frac{83}{2117682}\left(\frac{5}{7}\right)^5\\ +\frac{137}{50824368}\left(\frac{6}{7}\right)^5-\frac{1}{25412184}(1)^5\end{pmatrix}=0$$

$$C_{11}=\left(\frac{(1)^{11}}{11!}-\frac{1}{11!}.\begin{pmatrix}5.\left(\frac{1}{7}\right)^{11}-24.\left(\frac{2}{7}\right)^{11}+45.\left(\frac{3}{7}\right)^{11}\\ -40.\left(\frac{4}{7}\right)^{11}+15.\left(\frac{5}{7}\right)^{11}\end{pmatrix}\right)$$

$$-\frac{1}{6!}\begin{pmatrix}-\frac{1}{50824368}(0)^6+\frac{1}{12706092}\left(\frac{1}{7}\right)^6+\frac{209}{16941456}\left(\frac{2}{7}\right)^6\\ +\frac{23523}{25412184}\left(\frac{3}{7}\right)^6+\frac{8341}{50824368}\left(\frac{4}{7}\right)^6+\frac{83}{2117682}\left(\frac{5}{7}\right)^6\\ +\frac{137}{50824368}\left(\frac{6}{7}\right)^6-\frac{1}{25412184}(1)^6\end{pmatrix}=0$$

$$C_{12}=\left(\frac{(1)^{12}}{12!}-\frac{1}{12!}.\begin{pmatrix}5.\left(\frac{1}{7}\right)^{12}-24.\left(\frac{2}{7}\right)^{12}+45.\left(\frac{3}{7}\right)^{12}\\ -40.\left(\frac{4}{7}\right)^{12}+15.\left(\frac{5}{7}\right)^{12}\end{pmatrix}\right)$$

$$-\frac{1}{7!}\begin{pmatrix}-\frac{1}{50824368}(0)^7+\frac{1}{12706092}\left(\frac{1}{7}\right)^7+\frac{209}{16941456}\left(\frac{2}{7}\right)^7\\ +\frac{23523}{25412184}\left(\frac{3}{7}\right)^7+\frac{8341}{50824368}\left(\frac{4}{7}\right)^7+\frac{83}{2117682}\left(\frac{5}{7}\right)^7\\ +\frac{137}{50824368}\left(\frac{6}{7}\right)^7-\frac{1}{25412184}(1)^7\end{pmatrix}=0$$

$$C_{13}=\left(\frac{(1)^{13}}{13!}-\frac{1}{13!}.\begin{pmatrix}5.\left(\frac{1}{7}\right)^{13}-24.\left(\frac{2}{7}\right)^{13}+45.\left(\frac{3}{7}\right)^{13}\\ -40.\left(\frac{4}{7}\right)^{13}+15.\left(\frac{5}{7}\right)^{13}\end{pmatrix}\right)$$

$$-\frac{1}{8!}\begin{pmatrix} -\frac{1}{50824368}(0)^8+\frac{1}{12706092}\left(\frac{1}{7}\right)^8+\frac{209}{16941456}\left(\frac{2}{7}\right)^8 \\ +\frac{23523}{25412184}\left(\frac{3}{7}\right)^8+\frac{8341}{50824368}\left(\frac{4}{7}\right)^8+\frac{83}{2117682}\left(\frac{5}{7}\right)^8 \\ +\frac{137}{50824368}\left(\frac{6}{7}\right)^8-\frac{1}{25412184}(1)^8 \end{pmatrix}=\frac{31}{11719694698830720}$$

Hence the method is of order eight with Error constant $\left[\frac{31}{11719694698830720}\right]$

3.2. Consistency

Definition 3.1: The hybrid method (18) is said to be consistent if it has an order more than or equal to one i.e. $P \geq 1$. Therefore, the method is consistent ([15, 18, 20]).

3.3. Zero Stability

Definition 3.2: The hybrid method (18) is said to be zero stable if the first characteristic polynomial $\pi(r)$ having roots such that $|r_z| \leq 1 \quad and \quad if \quad |r_z| = 1$, then the multiplicity of r_z must not be greater than two ([15, 16, 23]). In order to find the zero-stability of hybrid method (18), we only consider the first characteristic polynomial of the method according to definition (3.2) as follows

$$\rho(z) = z - 5z^{\frac{1}{7}} + 24z^{\frac{2}{7}} - 45z^{\frac{3}{7}} + 40z^{\frac{4}{7}} - 15z^{\frac{5}{7}} \tag{21}$$

Setting equation (21) equal to zero and solving for *z* gives *z*=1, hence the method is zero stable.

3.4. Convergence

Theorem (2): Consistency and zero stability are sufficient condition for linear multistep method to be convergent. Since the method (7) is consistent and zero stable, it implies the method is convergent for all point ([15, 16, 19]).

3.5. Regions of Absolute Stability (RAS)

The absolute stability region of the new method is found according to ([16, 20, 21]). The region of its periodicity lies between (0,-24. 5). Hence It is A-stable in nature. See [20]

4. Numerical experiments

The method (18) was employed to solve (1) with the help of Taylor Series to provide starting values. The method is tested on two linear fifth order problems to test the accuracy of the proposed methods and our results are compared with the results obtained in the cited papers.

4.1. Numerical examples

The following problems are taken as test problems:
Example I: Consider a Linear fifth order problem

$$y^{v} = 32y + \cos x - 32\sin x,$$

$$y(0)=1, y'(0)=3, y''(0)=4, y'''(0)=7, y^{iv}(0)=16, h=0.1$$

$$y(x) = \sin x + e^{2x}$$

Source: [22]
Example II: Consider a Linear fifth order problem

$$y^{v} = 5y''' - 4y',$$

$$y(0)=3, y'(0)=-5, y''(0)=11, y'''(0)=-23, y^{iv}(0)=47, h=0.1$$

$$y(x) = 1 - e^{x} + 3e^{-2x}$$

Source: [24]

The following Notations were used in the tables

X-val	Value of the independent variables where numerical value is taken
Exact-solution	Exact solution at X-val
Computed-solution	Computed solution at X-val
Error	$\lvert Exact\ Solution - Computed\ Solution \rvert$

Table 1 Showing the exact solution, the computed solution and the absolute error in the developed method using Problem 1

x-values	Exact solution	Computed solution	Error in our Method
0.100000	1.321236174806998100	1.321236174806998100	0.000000e+000
0.200000	1.690494028436331700	1.690494028436331200	4.440892e-016
0.300000	2.117639007051848500	2.117639007051781400	6.705747e-014
0.400000	2.614959270801118200	2.614959270800171400	9.467982e-013
0.500000	3.197707367063248500	3.197707367057542000	5.706546e-012
0.600000	3.884759396131582500	3.884759396108925100	2.265743e-011
0.700000	4.699417654082365600	4.699417654012514800	6.985079e-011
0.800000	5.670388515294636300	5.670388515113154800	1.814815e-010
0.900000	6.832974374040428100	6.832974373623768500	4.166596e-010
1.000000	8.230527083738545400	8.230527082867489200	8.710561e-010

Table 2 Showing the exact solution, the computed solution and the absolute error in the developed method using Problem 2

x-values	Exact solution	Computed solution	Error in our Method
0.100000	2.551354841197985800	2.551354841197988500	2.664535e-015
0.200000	2.192229385028936500	2.192229385029126100	1.896261e-013
0.300000	1.905616687600363100	1.905616687602535600	2.172484e-012
0.400000	1.677666846316027400	1.677666846328145400	1.211808e-011
0.500000	1.497107663801695600	1.497107663847459900	4.576428e-011
0.600000	1.354770999642579600	1.354770999777845900	1.352662e-010
0.700000	1.243205588033408500	1.243205588371373100	3.379645e-010
0.800000	1.156360589866742400	1.156360590614063900	7.473215e-010
0.900000	1.089327004924157900	1.089327006430507400	1.506349e-009
1.000000	1.038126408538393000	1.038126411362454900	2.824062e-009

Table 3 Showing the comparison of error in the developed method and [22] using Problem 3

x-values	Error in our Method	Error in [22]
0.100000	0.000000e+000	2.000e-009
0.200000	4.440892e-016	9.000e-009
0.300000	6.705747e-014	1.000e-009
0.400000	9.467982e-013	3.900e-008
0.500000	5.706546e-012	3.700e-007
0.600000	2.265743e-011	1.966e-007
0.700000	6.985079e-011	9.373e-006
0.800000	1.814815e-010	3.632e-005
0.900000	4.166596e-010	1.203e-004
1.000000	8.710561e-010	3.523e-004

Table 4 Showing the comparison of error in the developed method and [24] using Problem 4

x-values	Error in our Method P=8, k=1	Error in [24] (2019), p=8, k=6
0.100000	2.664535e-015	3.108624468950438e-015
0.200000	1.896261e-013	2.398081733190338e-014
0.300000	2.172484e-012	6.894929072132072e-012
0.400000	1.211808e-011	1.453641651494309e-010
0.500000	4.576428e-011	1.681371930573050e-009
0.600000	1.352662e-010	1.226418588906597e-008
0.700000	3.379645e-010	6.574431687944582e-008
0.800000	7.473215e-010	2.808988770475196e-007
0.900000	1.506349e-009	1.009537685892070e-006
1.000000	2.824062e-009	3.165450833897410e-006

5. Conclusion

The optimization of One-step Hybrid Method proposed in this work was applied to solve fifth-order linear problems. This method has been shown to be efficient in terms of its applicability. The numerical Results of Problem 1 and Problem 2 are presented in Table 1 and Table 2. The comparison of this method with other existing ones namely [22] and [24] are shown in Table 3 and Table 4. It shows that the proposed method compares favorably and it's thus recommended for solution of linear fifth order ODEs

Compliance with ethical standards

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Disclosure of conflict of interest

The authors declare that they have no conflicts of interests.

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